**Algorithmic Game Theory** December 2, 2013

# Lecture 7

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## 1 Approximation algorithm for the Knapsack problem

In the previous lecture we saw the following 2-approximation algorithm for the Knapsack problem.

### 1.1 2-approximation algorithm

Denote $w_i$ player i weight and $b_i$ player i bid.
Sort the players according to $\frac{b_i}{w_i}$.
Well assume without loss of generality that $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq ... \geq \frac{b_n}{w_n}$.
The allocation will be the maximum between :
1. The player with maximal $w_i$.
2. Iterate over the players according to the previous sort, and allocate them in a greedy manner until exploiting all of the capacity.

**Theorem 1** *This algorithm achieves 2-approximation of the $OPT$.*

Is this implementable allocation rule?
According to Myerson's Lemma we should ask ourselves whether this is a monotone allocation rule.

**Theorem 2** *This algorithm implements monotone allocation rule.*

Fix the strategies for all players except one. If this player will raise her bid she obviously won't receive less.

**Fact 3** *We designed a 2-approximation mechanism for the knapsack problem.*

**Example 1** *The Knapsack problem has a FPTAS (Fully Polynomial Time Approximation Scheme). $\forall \epsilon > 0$ we have a $(1 + \epsilon)$-approximation algorithm which is polynomial in the*

*input size and $\frac{1}{\epsilon}$. The FPTAS allocation rule isn't monotone, hence we can deduce that it isn't implementable.*
*Recently it was proven that there is an implementable FPTAS.*

## 2 The revelation principle

Thus far we dealt with truthful mechanisms in dominant strategies. This is a strong requirement. We will try to see how this requirement limits our mechanisms.
Truthful mechanism have the following properties:
1. Each player has dominant strategy.
2. The strategy is direct revelation. This means that the player reveals her real value to the mechanism.

**Examples that weaken the mechanism**
1. Being truthful is beneficial in equilibrium, but it doesn't have to be a dominant strategy.
2. The dominant strategy is not telling the truth.

If we weaken the first requirement, for by requiring that we reach an equilibrium instead of implementability in dominant strategies, we might be able to achieve "better results" (In terms of performance, approximation, etc...).
According to the revelation principle there is no need to weaken the second requirement.

**Theorem 4** *For each mechanism $M$ in which each player has a dominant strategy, there exists a direct revelation mechanism $M'$ that achieves the same results.*

**Proof:** Each player $i$ has a private value $v_i$ and a dominant strategy $s_i(v_i)$ in mechanism $M$. We'll design a direct revelation mechanism $M'$ that will use $M$ in the following manner.

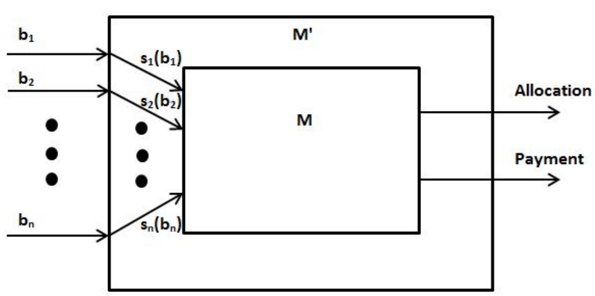



Figure 1: Mechanism M'

Mechanism $M'$ gets $b_1, b_2, ..., b_n$ as input and generate $s_1(b_1), s_2(b_2), ..., s_n(b_n)$ that are used

as $M$ input, $M'$ will return $M$ output as is.
Each player $i$ has a dominant strategy in mechanism $M'$, which is to revel her value $v_i$. Otherwise, it will contradict the fact that $s_i(v_i)$ is dominant strategy in mechanism $M$. ■

## 3 Multi-Parameter environments

In Multi-Parameter environments there are $n$ players, and a set of possible outcomes $\Omega$. Each player i has a private value $v_i(\omega)$ for each $\omega \in \Omega$.
For example, in a single item auction $|\Omega| = n+1$, since $\Omega$ will contain $n$ options that player $i$ won the auction, and an additional outcome that the auction had no winners.

## 4 Vickrey-Clarke-Groves (VCG) Mechanism

**Theorem 5** *For an allocation which maximizes the social welfare, there are always payments such that the mechanism is truthful in dominant stratagies.*

**Proof:** Vickrey, Clarke and Groves (VCG) proposed a method for these type of auctions which defines the allocation and payments, given the bids, in the following manner[1]:

1. Allocation Rule: $\omega^* = argmax_{\omega \in \Omega} \sum_{i=1}^{n} b_i(\omega)$.

2. Payments: $p_i(b) = max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\omega^*)$

One can notice that the allocation rule maximizes the social welfare, by definition. Also, the payment of the i'th player can be viewed as the "damage that this player caused society".

**Claim 6** *In the VCG mechanism, each player has a dominant strategy, and it is to be truthful (proof of this claim will obviously prove the theorem as well)*

**Proof:** Given a player i, and payments of the rest of the players $b_{-i}$:

$u_i(b) = v_i(\omega^*) - p_i(b) =$
$v_i(\omega^*) - max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) + \sum_{j \neq i} b_j(\omega^*) =$

[1]The following allocation and payment scehme is actually called the "Clarke-Pivot rule" and VCG is really a more general mechanism, in which: $p_i(b) = h_i(b_{-i}) - \sum_{j \neq i} b_j(\omega^*)$, $u_i(b) = v_i(\omega^*) + \sum_{j \neq i} b_j(\omega) - h_i(b_{-i})$. However, in our context we will refer to the Clarke-Pivot rule as VCG.

$\omega^* + \sum_{j \neq i} b_j(\omega^*) - max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)$

The second equalily holds by the definition of the payment ($p_i(b)$) in VCG.

The i'th player's objective is to maximize $u_i(b)$. The player does not affect the second part of the sum, $max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)$, so it will try to maximize the first part: $\sum_{i=1}^{n} v_i(\omega^*)$. The first part is exactly the definition of the social welfare, so in fact - this payment mechanism has aligned the intrest of the individual player i, with the social welfare itself! Therefore, in order to maximize it's own profit (and the social welfare as well), the players are better off being truthful. ■

Proof of the claim completes the proof of the theorem as well. ■

**Claim 7** *In the VCG mechanism:* $\forall i : p_i \geq 0$

**Proof:** Reminder: $p_i(b) = max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\omega^*)$. From maximality, $\forall \omega' \in \Omega : max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \geq \sum_{j \neq i} b_j(\omega')$, and this is true specifcally for $\omega' = \omega^*$. ■

**Claim 8** *In the VCG mechanism:* $\forall i : u_i \geq 0$

## 5 Examples of VCG Allocations and Payments

**Example 2** ***Single Product Auction***
*The result of VCG is the "Second Price Auction", namely:*
*Allocation: The product is given to the highest bidder ($i : max_i(b_i)$)*
*Payments: We saw that the payment is:* $p_i(b) = max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\omega^*)$ *(the Clarke-Pivot rule). Therefore, the buyer pays the price of the second highest bid:* $p_i = max_{j \neq i} b_j$*. For all other players, the payment is 0.*

**Example 3** ***K Identical Products***
*There are k identical products, and each player is interested in only one of these products. The value of the product to the i'th player is* $v_i$*:*
*Allocation: One product is allocated to each of the k highest bids.*
*Payments: Each player who is allocated a product causes the social damage they caused:* $p_i(b) = \sum_{j=1, j \neq i}^{k+1} b_j - \sum_{j=1, j \neq i}^{k} b_j = b_{k+1}$*. For all other players, the payment is 0.*

**Example 4** ***Procurement Auctions***
*We want to purchase 100 identical products from several potential providers. Each provider*

*has an infinite amount of the product, and a negative value (since their aim is to sell), for example:* $v_{Alice} = -17$, $v_{Bob} = -20$, $v_{Carol} = -30$.
*Allocation: The products are bought from the cheapest provider (lowest absolute value), in this case from Alice.*
*Payments: The value of the second bidder:* $p_A(b) = -20 - 0 = -20$. *For all other players, the payment is 0.*

**Example 5** ***Google Sponsored Search***
*Google sells advertisement slots on the right-hand toolbar. Given any search queries, Google sells k ordered advertisement slots. The k different slots are not identical (the higher ones are more valuable). The value of the different slots are determined by the amount of clicks ("pay per click").*
*CTR (click through rate): The probability of a user clicking on on the specific link.*
$\alpha_j$*: The CTR of the j'th slot. We will assume:* $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$.
*The value of the i'th player from each click is* $v_i$*, so the value of the j'th slot is* $v_i * \alpha_j$.
*Allocation: Let us order the players according to their valuations (descendingly). We will allocate the slots according to the order (the first slot to highest the player with the hights bid, k'th slot to k'th highest bid). All other players will not be allocated a spot.*
*Payments: The i'th player will pay:* $p_i(b) = (1/\alpha_i) * \sum_{j=1}^{k} b_{j+1}(\alpha_j - \alpha_{j+1})$*, which is the optimal welfare for the other players if player i was not participating minus the welfare of the other players from the chosen outcome. For all other players, the payment is 0.*

**Example 6** ***Bilateral Trade***
*There exist one salesman, one buyer and one item.*
*The item is worth value s for the salesman, and value b for the buyer*
***Allocation Rule***
*There exists a trade* $\iff b \geq s$
***Payment Method***
*If there is no trade, than obviously* $P_b = P_s = 0$.
*It there is a trade then:*

$$P_b = 0 - b = -b$$
$$P_s = s - 0 = s$$

*This means that in order to create a mechanism which promotes honesty, we must have subsidy.*

**Example 7** ***Public Goods***
*The government if facing the decision of whether to build a bridge.*
*The are for major sectors which benefit from building the bridge (marked A, B, C, D).*
*The benefit of each sector from building the bridge is:*

$V_A = 300$ $\quad$ $V_B = 400$
$V_C = 300$ $\quad$ $V_D = 200$

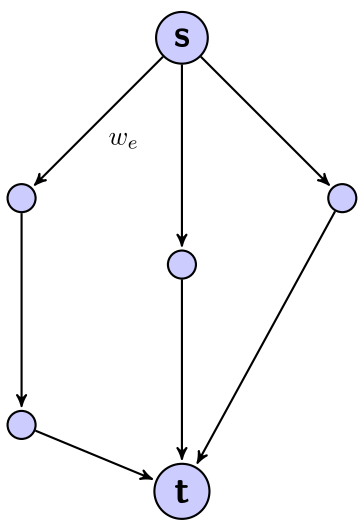
*The cost of building the bridge is:* $Cost = 1000$.
*The government would decide to build the bridge only if the cost of building it is smaller then the benefit of the people (represented here as the four sectors). We want to use a VCG mechanism (as defined above) for the government decision.*
***Allocation Rule***
$SW = \sum_{i=1}^{4} V_i - Cost$*, which means that the allocation rule is:*
*The bridge is built* $\iff \sum_{i=1}^{4} V_i \geq Cost$
***Payment Method***
*If the bridge is not built, than obviously* $P_b = P_s = 0$.
*It the bridge is built then:*

$$P_A = P_C = 0 - (900 - 1000) = 100$$
$$P_B = 0 - (800 - 1000) = 200$$
$$P_D = 0$$

*Again, we see that in order to create a mechanism which promotes honesty, we must have subsidy.*

**Example 8** ***Network routing***
*There exists a communication network, and we want to transfer a packet from node* $s$ *to node* $t$*. Denote by* $w_e$ *the cost of passing the packet through edge* $e$*.*
*In this "auction" the bidders are the edges of the network. The value of a route* $l$ *for each bidder is:*

$$V_e(l) = \begin{cases} -w_e, e \in l \\ 0, else \end{cases}$$




***Allocation Rule***
*We would send the packet through the route which brings* $\sum_{e \in l} w_e$ *to minimum, since* $SW = \sum_{e \in l} -w_e$.

***Payment Method***
*Say we choose route $l$, then the payment method is:*

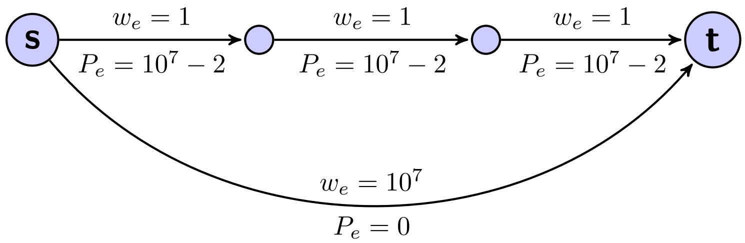
$$P_e = \begin{cases} \sum_{j\neq e} V_j(l') - \sum_{j\neq e} V_j(l) \ , \ if \ e \in l \\ 0, \ else \end{cases}$$

*Where $l'$ is the path that would have been chosen if player $i$ was not in the game. Where:*

$$\sum_{j\neq e} V_j(l') \equiv \ cost \ if \ w_e = \infty$$
$$\sum_{j\neq e} V_j(l*) \equiv \ cost \ if \ w_e = 0$$

*Which means that $P_e$ is the highest cost $e$ could has, that would still allow it to be on the shortest path from $s$ to $t$.*

***Example***




*This example shows us that the price of choosing a mechanism that promotes honesty can be very high.*

## 6 Combinatorial Auctions

**Definition 9** *A **Combinatorial Auction** is defined as:*
*There are $n$ players.*
*There are $m$ items in group $M$*
*Each player $i$ has a valuation function:*

$$V_i : 2^m \to \mathbb{R}$$
$$V_i(S), \forall S \subseteq M$$

**Assumption 10** $V_i(\emptyset) = 0$ *(Normalization)*

**Assumption 11** $V_i(S) \leq V_i(T), \forall S \subseteq M$ *(Free Disposal, Monotonicity)*

**Definition 12** *Two disjoint sets S, T are called **Substitute Sets** if:*
$V_i(S) + V_i(T) \geq V_i(S+T)$.

**Definition 13** *Two disjoint sets S, T are called **Complement Sets** if:*
$V_i(S) + V_i(T) \leq V_i(S+T)$.

**Example 9** ***Spectrum Auction by Regions***
*A spectrum auction is a process whereby a government uses an auction system to sell the rights (licences) to transmit signals over specific bands of the electromagnetic spectrum.*
*There are many types of spectrum auctions. In this example we talk about an auction where the rights sold are divided by bandwidth and by region. In this auction two bandwidths in the same region are **substitute items**, and two bandwidths in adjacent regions are **complement items**.*

**Example 10** ***Airport Time Slot Allocation Auction***
*A sealed-bid combinatorial auction is developed for the allocation of airport time slots to competing airlines. This auction procedure permits airlines to submit various contingency bids for flight-compatible combinations of individual airport landing or take-off slots.*