Algorithmic Game Theory	November 4, 2013

Lecture 4

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1 Submodular Functions

Definition 1 *A valuation function* $v : \mathcal{P}(M) \to \mathbb{R}^+$ *is* **subadditive**, *if for all packages* $S, T \subset M$ *such that* $S \cap T = \emptyset$, $v(S) + v(T) \geq v(S \cup T)$

Definition 2 *A valuation function* $v : \mathcal{P}(M) \to \mathbb{R}^+$ *is* **submodular**, *if for every package* T, *and every package* $S \subset T$, *for each product* $j \notin T$, $v(j|T) \leq v(j|S)$

Example 1 *The following function* v *is subadditive:*

$$v(R) = \begin{cases} 2 & |R| = 1 \\ 3 & |R| = 2 \\ 5 & |R| = 3 \end{cases}$$

However, it is not submodular: take a product x *and* $|S| = 1, |T| = 1$, *such that* $x \notin S \cup T$ *and* $S \neq T$,

$$\underbrace{v(S|x)}_{v(S\cup\{x\})-v(\{x\})=1} + \underbrace{v(T|x)}_{v(T\cup\{x\})-v(\{x\})=1} < \underbrace{v(S\cup T|x)}_{v(S\cup T\cup\{x\})-v(\{x\})=3}$$

2 Correlated and Coarse Correlated Equilibria

Definition 3 *A minimization game is* $(\lambda, \mu) -$ **smooth** *if for any two profiles,* s^* *and* s:

$$\sum_{i=1}^{n} c_i(s_i^*, s_{-i}) \leq \lambda \cdot cost(s^*) + \mu \cdot cost(s)$$

Theorem 4 *For each* (λ, μ)*-smooth game,* $PoA \leq \frac{\lambda}{1-\mu}$.

Proof: Same as the proof from the previous lecture. ■

Definition 5 *A joint distribution σ on strategy profiles is a* **correlated equilibrium** *(CE), if for each i, s_i, $s'_i \in S_i$:*

$$\mathop{\mathrm{E}}_{s\sim\sigma}[c_i(s)|s_i] \leq \mathop{\mathrm{E}}_{s_{-i}\sim\sigma_{-i}}[c_i(s'_i, s_{-i})|s_i]$$

Example 2 *Examine the following game:*

	C	*D*
C	*8,8*	*1,9*
D	*9,1*	*0,0*

Pure NE is at (C,D) and (D,C).

Every pure NE is a mixed equilibrium (ME), however is there any other ME?

The column player plays C with probability p, and D w.p. $1-p$.

If the row player plays C, the expectancy of the column player is $8p + (1-p)$

If the row player plays D, the expectancy of the column player is $9p$

$8p + (1-p) = 9p \to p = \frac{1}{2}$

An ME is reached when the both players play C or D with probability $\frac{1}{2}$.

In NE, social welfare is $\frac{9+1+9+1}{2} = 10$, and in ME it is $\frac{8+8+1+9+9+1+0+0}{4} = 9$.

Now, assume we want to play with the following probabilities:

	C	D
C	$\frac{1}{3}$	$\frac{1}{3}$
D	$\frac{1}{3}$	0

Note that this strategy profile is CE:

Assume the column player is told to choose strategy C:

- If he chooses C, then his expected utility is $\frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 1 = 4\frac{1}{2}$
- If he chooses D, then his expected utility is $\frac{1}{2} \cdot 9 + \frac{1}{2} \cdot 0 = 4\frac{1}{2}$

Similarly, assume the column player is told to choose strategy D, we can see that his expected utility if he chooses C and his expected utility if he chooses D are also equal.

Therefore, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ is CE, and the total expected welfare is $\frac{8+8+1+9+9+1}{3} = 12$.

Definition 6 *A joint distribution σ is* **Coarse Correlated Equilibrium** *if for every i and s'_i:*

$$\mathop{\mathrm{E}}_{s\sim\sigma}[c_i(s)] \leq \mathop{\mathrm{E}}_{s_{-i}\sim\sigma_{-i}}[c_i(s'_i, s_{-i})]$$

The following scheme represents the different types of equilibria we have learned so far:

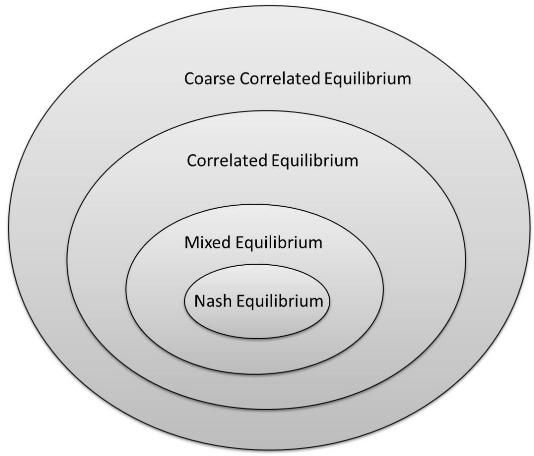


Figure 1: Relation between different equilibria

Our next goal is to try and bound the Price of Anarchy w.r.t CCE.

$$\frac{\underset{s\sim\sigma}{\mathrm{E}}[cost(s)]}{cost(s^*)} \leq ?$$

Theorem 7 [1] *Every boundary on PoA proven using the smoothness argument, is a valid boundary w.r.t ME, CE and CCE*

Proof:

$$\begin{aligned}
\underset{s\sim\sigma}{\mathrm{E}}[cost(s)] &= \underset{s\sim\sigma}{\mathrm{E}}[\sum_{i=1}^{n} c_i(s)] \quad (definition) \\
&= \sum_{i=1}^{n} \underset{s\sim\sigma}{\mathrm{E}}[c_i(s)] \quad (linearity\ of\ expectation) \\
&\leq \sum_{i=1}^{n} \underset{s\sim\sigma}{\mathrm{E}}[c_i(s_i^*, s_{-i})] \quad (\sigma\, is\, CCE) \\
&\leq \underset{s\sim\sigma}{\mathrm{E}}[\sum_{i=1}^{n} c_i(s_i^*, s_{-i})] \quad (linearity\ of\ expectation) \\
&= \sum_{i=1}^{n} \lambda \cdot cost(s^*) + \mu \cdot cost(s) \quad (Smoothness) \\
&= \lambda \cdot \underbrace{cost(s^*)}_{optimum} + \mu \cdot \underset{s\sim\sigma}{\mathrm{E}}[cost(s)]
\end{aligned}$$

[1] Roughgarden, Tim. "Intrinsic robustness of the price of anarchy." *Proceedings of the 41st annual ACM symposium on Theory of computing.* ACM, 2009.

$$\Rightarrow \frac{\mathop{\mathrm{E}}_{s\sim\sigma}[cost(s)]}{cost(s^*)} \leq \frac{\lambda}{1-\mu}$$

We saw that every NE is also a CCE (Figure 1), but there might be a CCE which is not NE, therefore $\underset{NE}{PoA} \leq \underset{CCE}{PoA}$ ■

In the previous lessons, we saw that in atomic selfish routing games $\underset{NE}{PoA} \leq 2.5$. We therefore also showed, using the smoothness argument, that $\underset{CCE}{PoA} \leq 2.5$.

We also saw a game where $\underset{NE}{PoA} = 2.5$, implying that this bound is tight. The smoothness argument does not increase the PoA even under CCE.

Definition 8 *A game in which the smoothness argument does not increase the boundary on PoA ($\underset{NE}{PoA} = \underset{CCE}{PoA}$) is called a* **tight game**.

3 Network Congestion Games

Definition 9 *A* **Network Congestion Game** *(NC) is defined as follows:*

- *Directed graph $G = (V, E)$ with a cost c_e for every $e \in E$.*
- *n players. Each player i has a source node $s_i \in V$ and a target node $t_i \in V$.*
- *The goal of player i: choose a path from s_i to t_i with the minimal cost.*
- *The set of strategies for player i is the set of possible routes from s_i to t_i.*
- *Given a strategy profile $\overrightarrow{p} = (p_1, \ldots, p_n)$, denote by p_i^j the jth vertex in path p_i. The graph generated $G_p = (V, E)$, where $V = \bigcup_i p_i^j$ and $E = \bigcup_i (p_i^{j-1}, p_i^j)$*
- *Denote by x_e the number of players who use edge e.*
- *The cost of each player who uses edge e is $\frac{c_e}{x_e}$.*
- *The cost for player i is defined as: $c_i(\overrightarrow{p}) = \sum_{e\in p_i} \frac{c_e}{x_e}$.*
- *The social welfare is defined as: $cost(p) = \sum_{i=1}^{n} c_i(p) = \sum_{e\in\bigcup_i p_i} c_e$.*

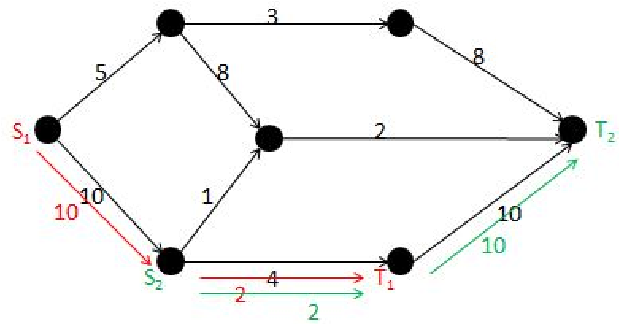


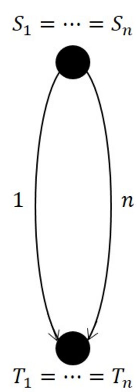
Figure 2: Example of an NC game

This is a *potential game*, and specifically, a *congestion game*. We can see that the following function is a potential function:

$$\Phi(p) = \sum_e \sum_{i=1}^{x_e} \frac{c_e}{i}$$

Our goal is to find a boundary for the PoA in NC game.

Example 3 *1-source, 1-destination NC game:*



In this game there are two Nash Equilbiria:

- *A good NE - if all players choose their route to be the edge with cost* 1.
- *A bad NE - if all players choose their route to be the edge with cost* n.

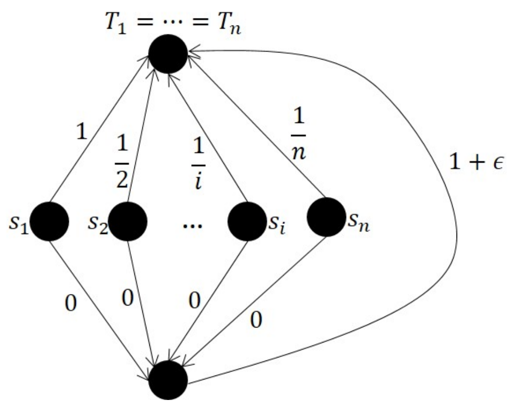
Therefore, for an NC, $PoA \geq n$.

Definition 10 *The price of stability for a set of games, G, is defined as:*

$$PoS = \max_{g \in G} \min_{s \in NE(G)} \frac{cost(s)}{cost(s^*)} = \frac{Best\,NE}{OPT}$$

where NE(g) is the set of Nash Equilibria for the game g, and s^* is an optimal strategy profile.

Example 4 *n-sources, 1-destination NC game:*



In this game there is a single NE: When each player i chooses to go through its direct edge to node t with cost $\frac{1}{i} \Rightarrow \sum_{i=1}^{n} \frac{1}{i} = H_n \approx log(n)$. We can also see that $OPT = 1 + \epsilon$ (and this is not an NE).

Therefore, $PoS \geq log(n)$.

Theorem 11 *In NC games, $\underset{NE}{PoA} \leq n$.*

Proof: Assume by way of contradiction, that there exists an NE p, such that $cost(p) > n \cdot cost(p^*)$. Hence, there exists a player i whose cost is higher than $cost(p^*)$, in contradiction to p being a NE, since player i can deviate to his own route in p^* and decrease her cost. ■

Theorem 12

$$PoS \leq H_n \approx log(n)$$

Proof: We saw that if for every profile p, there exist A, B such that $\frac{cost(p)}{A} \leq \Phi(p) \leq B \cdot cost(p)$, then $PoS \leq A \cdot B \Rightarrow \underbrace{cost(s)}_{Best\,NE} \leq A \cdot B \cdot \underbrace{cost(s^*)}_{OPT}$

Using this lemma, we show that for every p, $cost(p) \leq \Phi(p)$ and $\Phi(p) \leq H_n \cdot cost(p)$. This implies the aforementioned bound.

$$cost(p) = \sum_{e \in \bigcup_i p_i} c_e$$

$$\Phi(p) = \sum_e \sum_{i=1}^{x_e} \frac{c_e}{i} = \underbrace{\sum_e c_e}_{cost(p)} \cdot \underbrace{\sum_{i=1}^{x_e} \frac{1}{i}}_{H_{x_e} \leq H_n}$$

Therefore, $cost(p) \leq \Phi(p)$ and $\Phi(p) \leq H_n \cdot cost(p)$ ■