Algorithmic Game Theory	October 28, 2013
Lecture 3	
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1 Finding a Nash Equilibrium for Symmetric Selfish Routing in Polynomial Time

A *symmetric selfish routing* game is a selfish routing game where all players have the same source vertex s and destination vertex t.

Reminder: The *potential function* for a selfish routing game is given by $\sum_e \sum_{i=1}^{x_e} c_e(i)$.

Theorem 1 *For a symmetric selfish routing game, with monotonous edge cost, there is a polynomial-time algorithm that returns a NE.*

Proof: Given an instance of the routing game, we construct an instance of the MIN-COST-FLOW problem.

A MIN-COST-FLOW problem consists of a directed graph, with source vertex s and destination vertex t, where each edge has *capacity* α_e and *cost* c_e. The goal is to transport n transportation units from s to t, with minimal cost, while maintaining the capacity constraints.

We will use the following theorem:

Theorem 2 *There is a polynomial algorithm that solves MIN-COST-FLOW. Furthermore, for an instance with integer parameters, there is an integer optimal solution.*

Proof: MIN-COST-FLOW can be represented as a linear programming problem. ■

Given the original graph, for each edge e we will construct n parallel edges $e_1, \ldots, e_n$, each with capacity $\alpha_e = 1$, such that the cost of the edge e_i is $c_e(i)$.

The optimal solution cost is then $\sum_e \sum_{i=1}^{x_e} c_e(i)$, which is precisely the potential function for the selfish routing problem, and a minimum of the potential function corresponds to a NE.

■

2 The Price of Anarchy and Selfish Routing With Affine Costs

We want to know: how much inefficiency can we have in a NE? For example, in the Prisoner's Dilemma,

$$\begin{bmatrix} (n,n) & (n+1,0) \\ (0,n+1) & (1,1) \end{bmatrix}$$

the ratio between a player's utility in the NE $(1,1)$ and in the optimal solution (n,n), the solution with highest sum of player utilities, is unbounded.

To pose this question more clearly, we need to decide:

(1) What is the *social welfare function* that we want to optimize?

(2) How do we compare the result in the NE to the social optimum?

(3) Which definition of NE (mixed, pure) shall we use?

(4) If there is more than one NE, which one shall we choose?

Definition 3 *Given a family of games G, we define the* Price of Anarchy *(*PoA*) to be:*

$$PoA = \max_{g \in G} \max_{s \in NE(g)} \frac{cost(s)}{cost(s^*)}$$

where s^* is the optimal strategy profile and $NE(g)$ the set of NE's for the game g.

Remark $PoA \geq 1$. If $PoA = 1$ then any NE is as good as the optimum.

For this lecture, we will define:

Definition 4 $cost(s) = \sum_{i=1}^{n} c_i(s)$

and refer to PoA only relative to pure NE.

Remark $PoA_{pureNE} \leq PoA_{mixedNE}$.

Theorem 5 *For selfish routing games with affine cost functions, $c_e(x_e) = a_e x_e + b_e$, $PoA \le 2\frac{1}{2}$.*

Proof: $c_i(p) = \sum_{e \in p_i} c_e(x_e)$.

$cost(p) = \sum_{i=1}^{n} c_i(p) = \sum_{i=1}^{n} \sum_{e \in p_i} c_e(x_e)$.

We notice that:

$$cost(p) = \sum_{i=1}^{n} \sum_{e \in p_i} c_e(x_e) = \sum_{e \in E} c_e(x_e) \cdot x_e. \tag{1}$$

We will also use the following lemma:

Lemma 6 *(without proof) For every two integers $y, z \ge 0$,*

$$y \cdot (z+1) \le \tfrac{5}{3} \cdot y^2 + \tfrac{1}{3} z^2$$

and therefore, for each $a, b \ge 0$,

$$ay(z+1) + by \le \frac{5}{3} \cdot (ay^2 + by) + \frac{1}{3} \cdot (az^2 + bz). \tag{2}$$

We return to our proof. Let p be a NE and p^* an optimal solution, and let x and x^* be the corresponding load vectors. We want to show that $\frac{cost(p)}{cost(p^*)} \le 2\frac{1}{2}$.

$cost(p) = \sum_{i=1}^{n} c_i(p)$ (by definition of cost)

$\le \sum_{i=1}^{n} c_i(p_i^*, p_{-i})$ (since p is a NE)

Remark Here we used the fact that p is a NE in a minimal way: we use only the fact that there is no improvement when switching to p_i^*, without depending on p_{-i}.

When each player switches strategies, at most 1 is added to the load of each edge. From this and from (1) it follows that:

$\sum_{i=1}^{n} c_i(p_i^*, p_{-i}) \le \sum_{e \in E} a_e(x_e + 1) + b_e) \cdot x_e^*$

$\le \sum_{e \in E} \frac{5}{3}(a_e x_e^* + b_e)x_e^* + \sum_{e \in E} \frac{1}{3}(a_e x_e + b_e)x_e$ (from (2))

$= \frac{5}{3}cost(p^*) + \frac{1}{3}cost(p)$

$\Rightarrow cost(p) \leq 2\frac{1}{2} \cdot cost(p^*)$

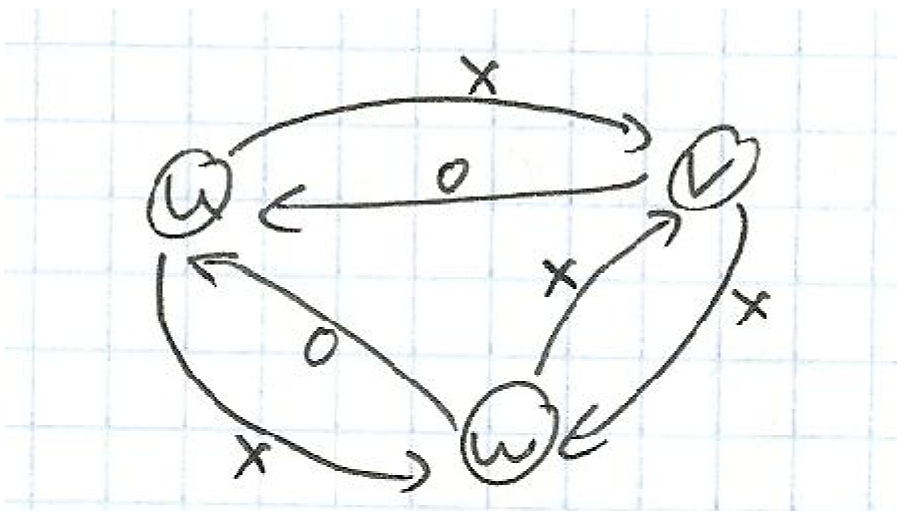
■

To review the proof we have just shown:

$$cost(p) \leq \sum_{i=1}^{n} c_i(p) \leq \sum_{i=1}^{n} c_i(p_i^*, p_{-i}) \leq \frac{5}{3}cost(p^*) + \frac{1}{3}cost(p).$$

Example 1 *An example showing the PoA of the routing game can be exactly 2.5:*

Player a has source v and destination w. Player b has source w and destination u. Player c has source u and destination v.



Edges labeled X have the following cost function: $C(x_i) = x_i$

Edges labeled O have the following cost function: $C(x_i) = x_i - 1$

$cost(s^*) = 2$ *for* s^* *being the routes* $u \to v, v \to w, w \to u$

$cost(s) = 5$ *for* s *being the routes* $u \to w \to v, v \to u \to w, w \to v \to u$

As you can see, s *is a NE and* $s/s^* = 2.5$*. Since we proved the PoA can't be any worse, the PoA of this game is 2.5.*

3 The Price of Stability

We can consider a case where we have no control over the players, who act to maximize their utility, but we can for example "set up" the game so the players start in an optimal NE.

Definition 7 *The price of stability for a set of games, G, is defined as:*

$$PoS = \max_{g \in G} \min_{s \in NE(g)} \frac{cost(s)}{cost(s^*)},$$

where NE(g) is the set of Nash Equilibria for the game g, and s^* is an optimal strategy profile.

The PoS of a set of games is the ratio between the cost of the best NE to the cost of the optimal solution. The PoA of a set of games is the ratio between the cost of the worst NE to the cost of the optimal solution. Therefore, PoA $\geq$ PoS.

Lemma 8 *Let G be a game with potential function ϕ, s.t. there exist $A, B \geq 1$, so that for any strategy profile s, $\frac{cost(s)}{A} \leq^1 \phi(s) \leq^2 B \cdot cost(s)$. Then $PoS(G) \leq A \cdot B$.*

Proof: Let s^* be an optimal solution, and let s an the absolute minimum of ϕ.

$cost(s) \leq A \cdot \phi(s)$ (from [1])

$\leq A \cdot \phi(s^*)$ (s^* is optimal)

$\leq A \cdot B \cdot cost(s^*)$ (from [2])

$\Rightarrow \frac{cost(s)}{cost(s^*)} \leq A \cdot B$

$\Rightarrow PoS \leq A \cdot B$

■

Theorem 9 *In routing games with affine cost functions $C(X_i) = aX_i + b$, $PoS \leq 2$.*

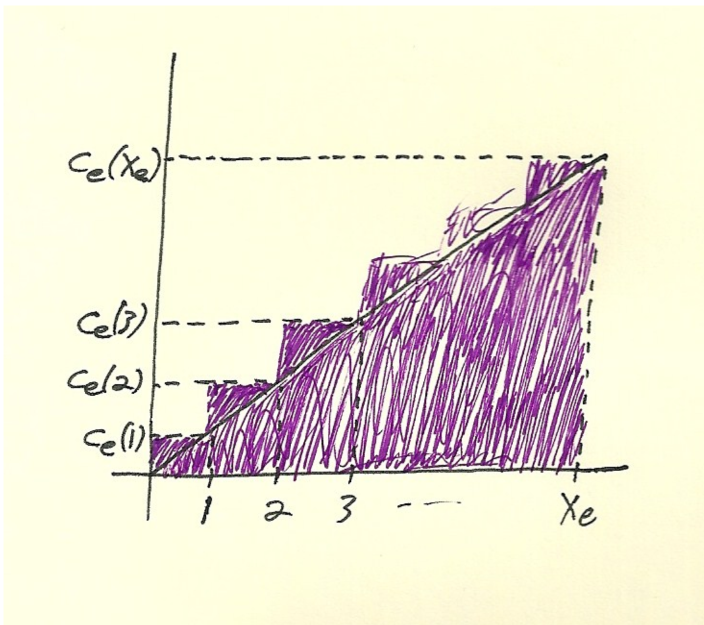
Proof: $cost(s) = \sum_{e \in E} X_e \cdot C_e(X_e)$

$\phi(s) = \sum_{e \in E} \sum_{i=1}^{X_e} C_e(i)$

$=> \phi(s) \leq cost(s)$.

This is the second predicate for our previous lemma, with B = 1. To use the lemma, we need only prove the first predicate holds. We will do so for A = 2, that is, $\phi(s) \geq cost(s)/2$.

For each $e \in E$:



With every player that uses the edge, cost(s) goes up by the area of the rectangle, while $\phi(s)$ goes up by at least half the area. Therefore, $\phi(s) \geq cost(s)/2$. ■

4 Simultaneous Second Price Auction

We now consider a second-price auction with n participants and m products. Each player i has a *valuation function* $v_i(T)$ that assigns a value for each $T \subseteq [m]$.

We assume that the valuation functions are *sub-modular*:

$\forall T \subseteq S, j \notin S, \quad v_i(T \cup \{j\}) - v_i(T) \geq v_i(S \cup \{j\}) - v_i(S)$.

Each player i offers a bid b_{ij} for each product j, and each product is then sold separately as in a second-price auction.

We assume that there is *no over-bidding*:

$\forall T, i, \quad \sum_{j \in T} b_{ij} \leq v_i(T)$.

Let b be a strategy profile, where $b = (b_1, \ldots, b_n)$, $b_i = (b_{i1}, \ldots, b_{im})$.

Notations:

$x_i(b)$ - the set of products which player i wins under the profile b.

$b_j^{(2)}$ - the second-highest bid for product j.

It follows that $p_i(b) = \sum_{j \in x_i(b)} b_j^{(2)}$ is the payment for player i under the profile b.

The *utility function* for player i is $u_i(b) = v_i(x_i(b)) - p_i(b)$.

Our target function (*social welfare function*) is $w(b) = \sum_{i=1}^{n} v_i(x_i(b))$.

Theorem 10 *In a simultaneous second price auction, $PoA \leq 2$.*

Proof: Let b^* be an optimal strategy profile, and let b be a Nash Equilibrium.

$w(b) = \sum_{i=1}^{n} v_i(x_i(b)) \geq \sum_{i=1}^{n} u_i(b_i, b_{-i})$ (the payments are non-negative)

$\geq \sum_{i=1}^{n} u_i(b_i^*, b_{-i})$ (b is a Nash Equilibrium)

$\geq w(b^*) - w(b)$. (we will prove this shortly)

It follows that $w(b) \geq \frac{1}{2} w(b^*)$.

We will now prove that $\sum_{i=1}^{n} u_i(b_i, b_{-i}) \geq w(b^*) - w(b)$.

Let $T^* = (T_1^*, \ldots, T_n^*)$ be an optimal partition of products, that maximizes $\sum_{i=1}^{n} v_i(T_i^*)$. We now want to find bids b_{ij}^* that result in that optimal partition.

We assume without loss of generality that the player i recieves the products $1, \ldots, d$ in T_i^*. We define:

$b_{ij}^* = v_i(1, \ldots, j) - v_i(1, \ldots, j-1)$ for all $j \leq d$,

$b_{ij} = 0$ for all other j.

Then $\sum_{j \in T_i^*} b_{ij}^* = v_i(T_i^*)$. (telescoping series)

It follows from sub-modularity that for all T, $\sum_{j \in T} b_{ij}^* \leq v_i(T)$.

Notation: we denote by S the set of products from T_i^* that player i wins under the strategy profile (b_i^*, b_{-i}). $S \subseteq T_i^*$.

$u_i(b_i^*, b_{-i}) = v_i(S) - \sum_{j \in S} \max_{l \neq i} b_{lj}$ (from the definition of second price auction)

$\geq \sum_{j \in S} (b_{ij}^* - \max_{l \neq i} b_{ij})$ (sub-modularity)

$\geq \sum_{j \in T_i^*} (b_{ij}^* - \max_{l \neq i} b_{ij})$ (if $j \in T_i^* \setminus S$ then player i didn't win j, therefore $b_{ij}^* \leq \max_{l \neq i} b_{ij}$)

$\geq v_i(T_i^*) - \sum_{j \in T_i^*} \max_{l=1,\ldots,n} b_{lj}$. ($v_i(T_i^*) = \sum_{j \in T_i^*} b_{ij}^*$, $\max_{l \neq i} b_{lj} \leq \max_{l=1,\ldots,n} b_{lj}$)

We have shown that $u_i(b_i^*, b_{-i}) \geq v_i(T_i^*) - \sum_{j \in T_i^*} \max_{l=1,\ldots,n} b_{lj}$. We sum these inequalities over all i:

$\sum_{i=1}^{n} u_i(b_i^*, b_{-i}) \geq \sum_{i=1}^{n} v_i(T_i^*) - \sum_{j=1}^{m} \max_{l=1,\ldots,n} b_{lj}$ (T^* is a partition of all products)

$\geq w(b^*) - \sum_{i=1}^{n} \sum_{j \in T_i} b_{ij}$ (T_i is the set of products won by player i in the profile b)

$\geq w(b^*) - \sum_{i=1}^{n} V_i(T_i)$ ($\sum_{j \in T_i} b_{ij} \leq V_i(T_i)$)

$= w(b^*) - w(b)$.

We have shown that $\sum_{i=1}^{n} u_i(b_i^*, b_{-i}) \geq w(b^*) - w(b)$.

■

Let's compare the structures of the two proofs we have seen:

Auction	Selfish Routing
$w(b) \geq \sum_i u_i(b)$	$cost(p) \geq \sum_i c_i(p)$
$\geq \sum_i u_i(b_i^*, b_{-i})$ (Nash Equilibrium)	$\geq \sum_i c_i(p_i^*, p_{-i})$ (Nash Equilibrium)
$\geq 1 * w(b^*) - 1 * w(b)$ (smoothness, $\mu = 1, \lambda = 1$)	$\geq \frac{5}{3}cost(p^*) + \frac{1}{3}cost(p)$ (smoothness, $\mu = \frac{5}{3}, \lambda = \frac{1}{3}$)
$\Rightarrow \frac{w(b)}{w(b^*)} \geq \frac{\lambda}{1+\mu} = \frac{1}{2}$	$\Rightarrow \frac{cost(p)}{cost(p^*)} \leq \frac{\lambda}{1-\mu} = 2\frac{1}{2}$

Smoothness is a property of the game, that gives us a general technique for bounding PoA.
We will discuss it further in the next lecture.