Lecture 2

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1 Min-max theorem

Theorem 1 *Let $x = \{x_1, ..., x_n\}$ and $y = \{y_1, ..., y_m\}$ be mixed strategies (e.g. $\sum_{i=1}^{n} x_i = 1$ and $\sum_{i=1}^{m} y_i = 1$).*

$$\max_x \min_j \sum_i x_i a_{ij} = \min_y \max_i \sum_j y_j a_{ij} = v$$

v is called the value of the game. The strategies (x, y) which provide the max *and* min *respectively are a Nash Equilibrium (NE).*

Proof: The row player tries to solve the following optimization problem (OPT_1):

$$\max_x \min_j \sum_i x_i a_{ij}$$
$$\text{s.t.} \quad \sum_i x_i = 1$$
$$\forall i : x_i \geq 0$$

This problem is equivalent to the following linear programming problem (OPT_2):

$$\max c$$
$$\text{s.t} \quad \sum_i x_i a_{ij} \geq c$$
$$\sum_i x_i = 1$$
$$\forall i : x_i \geq 0$$

In a similar way, the column player tries to solve the following optimization problem (OPT_4):

$$\min_y \max_i \sum_j y_j a_{ij}$$
$$\text{s.t.} \quad \sum_j y_j = 1$$
$$\forall j : y_j \geq 0$$

And this problem is equivalent to the following linear programming problem (OPT_5):

$$\begin{aligned} &\min d \\ &\text{s.t} \quad \sum_j y_j a_{ij} \le d \\ &\qquad \sum_j y_j = 1 \\ &\qquad \forall j : y_j \ge 0 \end{aligned}$$

Let us introduce an optimization problem OPT_3:

$$\begin{aligned} &\min_x \sum_i x_i \\ &\text{s.t} \quad \forall j \sum_i x_i a_{ij} \ge 1 \\ &\qquad \forall i : x_i \ge 0 \end{aligned}$$

Claim 2 $OPT_2 = \frac{1}{OPT_3}$.

Proof: $OPT_2 \le \frac{1}{OPT_3}$: Let x be a solution for OPT_2, then $\frac{x}{c}$ is a feasible solution for OPT_3 with the value $\frac{1}{c}$.
$OPT_2 \ge \frac{1}{OPT_3}$: Let x be a solution for OPT_3, then xc is a feasible solution for OPT_2, with the value c.
■

Similarly we can define OPT_6:

$$\begin{aligned} &\max_y \sum_j y_j \\ &\text{s.t} \quad \forall i \sum_j y_j a_{ij} \ge 1 \\ &\qquad \forall j : y_j \ge 0 \end{aligned}$$

And $OPT_5 = \frac{1}{OPT_6}$.

From the duality principle, $OPT_3 = OPT_6$, therefore $OPT_2 = OPT_5$ and $OPT_1 = OPT_4$.
■

Corollary 3 *In any zero-sum game, it is possible to commute a NE in polynomial time.*

2 Yao's principle

Theorem 4 *In order to get a lower bound for a random algorithm's runtime, it is sufficient to get a lower bound for the best deterministic algorithm's runtime, where its performance is measured over some input distribution.*

Proof: Consider a game of two players: The "Algorithms" player, and the "Inputs" player; where a_{ij} is the runtime of algorithm j for input i. Then

$$\min_x \max_i \underbrace{E[A_x, I_i]}_{\text{distribution over algorithms}} \leq \max_y \min_j \underbrace{E[A_j, I_y]}_{\text{distribution over inputs}} \tag{1}$$

■

Theorem 5 *Given a two-player game, in which each player has m strategies, the problem of telling whether there is an equilibrium such that the sum of the values is greater than 2, is* NP_{hard}.

Proof: We show a reduction from the Balanced-bi-clique problem: Given a bi-partite graph $G = (V_U, V_D, E)$ and k, are there sets $U, D, |U| = |D| = k$ such that every node in U is connected to every node in D.

Consider the following game: each player chooses a node from either V_U or V_D.
If player 1 chooses a node $v \in V_U$ and player 2 chooses $u \in V_D$, the utility is $(1, 1)$ if $(u, v) \in E$, otherwise the utility is $(0, 0)$.
If player 1 chooses from V_D and player 2 from V_U, the utility is $(0, 0)$.
If both player 1 and player 2 choose from V_U, then the utility is $(0, 0)$ unless they choose the same node, in which case the utility is $(-k, k)$.
If both player 1 and player 2 choose from V_D, then the utility is $(0, 0)$ unless they choose the same node, in which case the utility is $(k, -k)$.

	V_U			V_D		
V_U	(-k,k)	(0,0)	(0,0)	(1,1) if connected (0,0) otherwise		
	(0,0)	...	(0,0)			
	(0,0)	(0,0)	(-k,k)			
V_D	(0,0)			(k,-k)	(0,0)	(0,0)
				(0,0)	...	(0,0)
				(0,0)	(0,0)	(k,-k)

Claim 6 *If there is a k-bi-clique in the graph, then there is a NE in the game, such that the sum of values is ≥ 2.*

Proof: Denote the k-bi-clique by $V'_U \subseteq V_U, V'_V \subseteq V_D$ (so $|V'_U| = |V'_D| = k, V'_U \times V'_D \subseteq E$). We show that the strategy where player 1 chooses nodes V'_U and player 2 chooses nodes from V'_D, both choose their node with probability $\frac{1}{k}$ for each node, is a NE, thus creating a utility of $(1, 1)$.
It is a NE because Player 1, by choosing a node from V_D, can only win k with probability $\frac{1}{k}$, providing an expected value of 1, and can win 0 by choosing nodes from $V_U \setminus V'_U$.
Similarly, Player 2 cannot benefit from choosing a node from V_U, or from changing the nodes within V_D. Choosing nodes from V'_U and V'_D respectively will provide each player a value of 1, regradless of the distribution of the nodes. ■

Claim 7 *If there is a NE such that the sum of the values is ≥ 2, then there is a k-bi-clique.*

Proof: First, we notice that if the utility is ≥ 2, then only values in the upper-right corner can have positive probabilities, since they contribute a value of 2, while all other options contribute 0. Then, we can notice that the probabilities of Player 1 cannot be greater than $\frac{1}{k}$ for each node, because then Player 2 can benefit from choosing from V_U.
Similarly, Player 2's probabilities are also at most $\frac{1}{k}$ for each of his nodes.
The value is ≥ 2, and all the positive probabilities are in the $V_U \times V_D$ square - this can only be achieved if all the selected nodes are connected. Since all the probabilities are at most $\frac{1}{k}$, there is at least a k-clique. ■

This concludes the proof of theorem 5. ■

3 The Aloof Game

Let $G = (V, E)$ be an undirected graph, V the group of players, and $S_i = \{white, blue\}$ the set of strategies.
The value for player i is: $u_i(s_i, \cdots, s_n) = |\{j : (i,j) \in E, s_i \neq s_j\}|$: the number of i's neighbors that do not match i's color.

Questions:

1. Is there a NE with pure strategies? Is it unique?
2. If so, is there a natural process to find it?
3. If so, does it converge in polynomial time?

Theorem 8 *The Aloof game has a pure NE*

To prove theorem 8, we require the following two claims.

Claim 9 *The optimal solution to the MAX-CUT problem is a pure NE.*

Proof: Let (S, T) be an optimal cut to MAX-CUT problem of the graph of the Aloof game $G = (V, E)$.
Note the size of the cut $|(S,T)| = |(S \times T) \cap E|$. Without loss of generality, the (S, T) cut is equivalent to a strategy s such that:

$$s_i = white \iff i \in S$$

For any player $i \in S$, construct a new cut (S', T') where $S' = S \setminus \{i\}$ and $T' = T \cup \{i\}$. The size of the (S', T') cut is:

$$|(S', T')| = |(S,T)| + u_i(s') - u_i(s),$$

where $s' = blue$.
Since (S, T) is an optimal cut then $|(S', T')| \leq |(S,T)|$, Therefore, $u_i(s') - u_i(s) \leq 0$. Since there are only 2 strategies for player i, s_i is a best-response for s_{-i}.
The proof is similar for any player $i \in T$. This proves that s is a pure NE. ■

Greedy approximation for MAX-CUT - Consider the following algorithm:

1. Start with an arbitrary coloring
2. If there is a change that increases the cut, perform that change.
3. Repeat the previous step (while possible).

Claim 10 *The algorithm above will stop in a NE equilibrium.*

Proof: The algorithm necessarily terminates because the cut always improves, and the number of iterations is at most $|E|$. Upon termination, we will have a coloring s. Since the algorithm stopped, s must satisfy the claim that there are no edges that would improve the cut by switching color. Similarly to the previous claim, for any player i we can construct a new cut (S', T') such that i switched sides, which implies that $|(S', T')| \leq |(S, T)|$, and then again $u_i(s') - u_i(s) \leq 0$. ■

Claim 11 *The algorithm above a provides 2-approximation to the optimal solution.*

Proof: $\forall i \in V : C_i = i$'s cut, $N_i = i$'s neighbors. $|C_i| \geq \frac{|N_i|}{2}$, otherwise, Player i can improve. By summing over all of the players, we get an approximate cut of $\sum\_i \in V \frac{|C_i|}{2} = \sum \frac{|N_i|}{4}$ and by the standard bound: $|\text{MAX-CUT}| \leq E = \sum \frac{|N_i|}{2}$, we get that the approximate MAX-CUT is at most 2 times smaller than the MAX-CUT. ■

Question 4: What can be said for the weighted problem of the game (where each edge e has a weight $w_e \geq 0$)?

Definition 12 *A game is called a potential game, if there exists a function $\phi : S_1 \times \cdots \times S_n \to \mathbb{R}$ such that for every player i, for every $x_i, x_i' \in S_i$ and for every $x_{-i} \in S_{-i}$ it holds that $u_i(x_i, x_{-i}) > u_i(x_i', x_{-i}) \iff \phi(x_i, x_{-i}) > \phi(x_i', x_{-i})$.*

Theorem 13 *Every final potential game has a pure NE.*

Proof: Every local maximum of ϕ is a NE. ■

Definition 14 *A game is called an exact potential game, if there exists a function $\phi : S_1 \times \cdots \times S_n \to \mathbb{R}$ such that for every player i, for every $x_i, x_i' \in S_i$ and for every $x_{-i} \in S_{-i}$ it holds that $u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = \phi(x_i, x_{-i}) - \phi(x_i', x_{-i})$.*

Theorem 15 *If every locally improving algorithm always terminates, then the game is a potential game.*

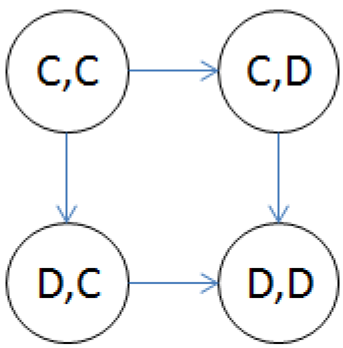
Example 1 *Consider the prisoner's dilemma - with the potential function: $\phi(C, C) = 0, \phi(C, D) = \phi(D, C) = 1, \phi(D, D) = 2$.*

Player1 \ Player2	C	D
C	(3,3), Φ=0	(5, 0) , Φ=1
D	(0, 5) , Φ=1	(1, 1) , Φ=2

Definition 16 *An improvement graph is a directed graph where:*

- *There is one node for each strategy profile.*
- *There is an edge $a \to b$ iff node b differs from a by the change of a single player's strategy and, for that player, the strategy of b dominates (diverting from atob would be beneficial).*

Improvement graph for the prisoner's dilemma:



This is an acyclic graph, thus every locally improving algorithm will converge.

4 Selfish Routing

Definition 17
A selfish routing *game is defined as follows:*

- $G = (V, E)$ *a directed graph.*
- n *players.*
- $\forall i \in [n]$ *there is a source node* s_i *and a destination node* t_i.
- *Every edge has a cost function* $c_e(x_e) \geq 0$ *on the number of players using the edge* e.
- P_i *a set of paths from* s_i *to* t_i *for player* i.

- $p_i \in P_i$, *the chosen path of player* i.
- $\overline{p} = (p_1, \cdots, p_n)$ *the vector of chosen paths.*
- $c_i(p) = \sum_{e \in p_i} c_e(x_e(p))$ *is the cost of player* i *(for simplicity we use* $x_e = x_e(p)$*).*

Theorem 18 *Every selfish routing game has a pure NE.*

Proof: We show that every selfish routing game is an exact potential game.

$$\phi(p) = \sum_{e \in E} \sum_{i=1}^{x_e(p)} c_e(i)$$

Suppose a player i diverges from path p_i to path p'_i.
We need to prove that $\phi(p_i, p_{-i}) - \phi(p'_i, p_{-i}) = c_i(p_i, p_{-i}) - c_i(p'_i, p_{-i})$,
what happens to the cost function $c_i(\cdot)$?

There are 3 types of edges:

- $e \in p_i \setminus p'_i$
- $e \in p'_i \setminus p_i$
- $e \in p_i \cap p'_i$

$$c_i(p_i, p_{-i}) = \sum_{e \in p_i} c_e(x_e)$$

Denote $p = (p_i, p_{-i})$ and $p' = (p'_i, p_{-i})$.

Type of edge	Change in c_i	Change in ϕ
$e \in p_i \cap p'_i$	0	0
$e \in p_i \setminus p'_i$	$-c_e(x_e(p))$	$-c_e(x_e(p))$
$e \in p'_i \setminus p_i$	$c_e(x_e(p)+1)$	$c_e(x_e(p)+1)$

$$c_i(p_i, p_{-i}) - c_i(p'_i, p_{-i}) = \sum_{e \in p_i \setminus p'_i} c_e(x_e) - \sum_{e \in p'_i \setminus p_i} c_e(x_e + 1)$$

$$\phi(p) = \sum_{e \in E} \sum_{i=1}^{x_e} c_e(i)$$

$$\phi(p_i, p_{-i}) - \phi(p'_i, p_{-i}) = c_i(p_i, p_{-i}) - c_i(p'_i, p_{-i}).$$

■

5 Congestion games

Definition 19 *Define a congestion game as follows: E - a set of resources, n players. Every player has $S_i \subset 2^E$ strategies, every resource has the cost $c_e(x_e)$, where x_e is the number of players using the resource e. The cost function of player i is $C_i(s) = \sum_{e \in S_i} c_e(x_e)$.*

Theorem 17 holds for any congestion game as well:

$$\phi(\underbrace{s}_{\text{strategy profile}}) = \sum_{e \in E} \sum_{i=1}^{x_e} c_e(i) \tag{2}$$

Corollary 20 *Every congestion game is an exact potential game, and therefore has a pure NE.*