Algorithmic Game Theory January 13, 2014

Lecture 13

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1 Prior-free Mechanisms

Let there be digital goods with the values $v = (v_1, ..., v_n)$ where v_i is the value of player i for a unit of the digital goods being auctioned. We want to maximize the revenue. We will compare the revenue of our mechanism to the optimal revenue we can get from the envy-free mechanism.

We define the following order on the player valuations $v_{(1)} \geq v_{(2)} \geq ... \geq v_{(n)}$, and the *Envy Free Optimal* result $EFO = max_{1 \leq i \leq n} i v_{(i)}$. There is no mechanism that gives a sub-linear approximation with respect to $EFO(v)$.

We define the following $EFO^{(2)}(v) = max_{i \geq 2} i v_{(i)}$. We would like to design a truthful mechanism that gives a good approximation with respect to $EFO^{(2)}$, for every valuation vector $\overrightarrow{v}$. Assume we want to get revenue R from every input $\overrightarrow{v}$ for $EFO(v) \geq R$. A mechanism designed to provide a target revenue R is called a *Profit Extractor*.

The Mechanism: Given a target R and a profile $\overrightarrow{v}$, find the largest k such that $v_{(k)} \geq R/k$, and sell to the k players with price R/k each (if k exists). If k doesn't exist, no one gets anything.

$$\underbrace{v_{(1)} \geq v_{(2)} \geq ... \geq v_{(k)}}_{k} \geq ... \geq v_{(n)}$$

$$\frac{R}{k} k = R$$

Theorem 1 *This mechanism is truthful.*

Proof: Note that the price can only increase in each iteration. If $EFO(v) \geq R$ then $R \leq EFO(v) = k v_{(k)}$. We get $R/k \leq v_{(k)}$. The mechanism will find some k that gives us R. If $EFO(v) < R$ then there's no k such that $R/k \leq r_{(k)}$, therefore the profit of the mechanism will be 0. ■

Random Sampling Profit Extraction Auction

1. Partition the players into two groups S',S'' by repeatedly flipping a fair coin.
2. Compute $R' = EFO(v_{S'})$ and $R'' = EFO(v_{S''})$.
3. Apply the profit extractor to S' with target R'' and onto S'' with target R'.

This mechanism is truthful and its profit is $min(R', R'')$.

Theorem 2 *The random mechanism is a 4-approximation to* $EFO^{(2)}$*.*

To prove Theorem 2 we first show Lemma 3:

Lemma 3 *Flipping* $k \geq 2$ *fair coins gives* $E[min(\#heads, \#tails)] \geq \frac{k}{4}$*.*

Proof of Lemma 3: Let M_i be a random variable for $\min\{\#heads, \#tails\}$ after a fair coin is tossed i times.

$$\begin{aligned} E[M_1] &= 0 \\ E[M_2] &= \tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 1 = \tfrac{1}{2} \\ E[M_3] &= \tfrac{1}{4} \cdot 0 + \tfrac{3}{4} \cdot 1 = \tfrac{3}{4} \end{aligned}$$

We show a general bound on $E[M_i]$ for $i > 3$:
Let X_i be a random variable representing the difference of $\min\{\#heads, \#tails\}$ between two consecutive tosses:

$$X_i = M_i - M_{i-1}$$

We calculate $E[X_i]$:
Case 1: i is even

$$E[X_i] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Case 2: i is odd ($E[X_i] \geq 0$)
For $i = 3$:

$$E[M_2] = \frac{1}{2}$$

$$E[M_3] = \frac{3}{4}$$

$\Rightarrow E[X_3] = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$
So we get:

$$E[M_k] = \sum_{i=1}^{k} E[X_i] \geq 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + 0 + \frac{1}{2} + ... = \frac{1}{4} + \frac{\lfloor \frac{k}{2} \rfloor}{2} \geq \frac{k}{4}$$

■

Proof of Theorem 2: Let us assume that $EFO^{(2)}$ sells to $k \geq 2$ buyers at price p.
Then: $Rev[EFO^{(2)}(v)] = k \cdot p$
Of the k buyers, let k' be the number of buyers in S' and k'' be the number of buyers in S''.
We know that: $R' \geq k' \cdot p$ and $R'' \geq k'' \cdot p$.
So, we get that:

$$\frac{Rev[our\ mechanism]}{Rev[EFO^{(2)}(v)]} = \frac{E[\min\{R', R''\}]}{k \cdot p} \geq \frac{E[\min\{k' \cdot p, k'' \cdot p\}]}{k \cdot p} = \frac{E[\min\{k', k''\}]}{k} \geq \frac{1}{4}$$

where the final inequality is derived from the lemma.
■

2 Equilibrium Prices

We are given m items and n buyers.
How should we determine the prices of the items?
Each buyer i has a valuation function v_i,

$$v_i(S) \in \mathbb{R}^+ \ \forall S \subseteq [m]$$

Standard assumptions:
normalization: $v_i(\phi) = 0$
monotony: $v_i(S) \leq v_i(T) \forall S \subseteq T$

Definition 4 *Given a vector $p = (p_1, ..., p_n)$ of item prices, we define the utility for buyer i on set S as:*

$$u_i(S) = v_i(S) - \sum_{j \in S} p_j$$

3 Walrasian Equilibrium

Our goal is to maximize the social welfare $SW = \sum_i v_i(S_i)$, with the given prices $p = (p_1, p_2, ..., p_n)$ and an allocation $S_1, S_2, ..., S_n$ where the following holds:

1. Each player i receives a set of products S_i in his *demand set* (collection of all sets that maximize the utility).
2. For every product that is not allocated $j \notin \bigcup_{i \in N}$, the price $p_j = 0$.

Configuration LP

Given a random variable $X_{i,S} = \begin{cases} 1, & S \to i \\ 0, & otherwise \end{cases}$

$$max = \sum_{i \in N, S \subseteq M} X_{i,S} v_i(S)$$

s.t.:

$$\sum_{i \in N, S_{i,j} \in S} X_{i,S} \leq 1, \forall j \in M$$
$$\sum_{S \subseteq M} X_{i,S} \leq 1, \forall i \in N$$
$$X_{i,S} \in \{0,1\} \ \forall i \in N, \ S \subseteq M$$

Problems:

- The number of players is exponential.
- This program is an *Integer Program.*

We will do *Linear Programming Relaxation*, $X_{i,S}$.

Theorem 5 *If there exists a* Walrasian Equilibrium *WE then it maximizes the social welfare. Moreover, it maximizes the SW even over the fractional solutions.*

Proof: If $(p_1^*, ..., p_m^*)$ and $(S_1^*, ..., S_n^*)$ is a WE, then for all the feasible LPR solutions (including fractional) $\left\{X_{i,S}^*\right\}_{i,S}$, it holds that:

$$\underbrace{\sum_{i=1}^{n} v_i(S_i^*)}_{\text{SW of WE}} \geq \underbrace{\sum_{i \in N, S \subseteq M} X_{i,S}^* v_i(S)}_{\text{SW of the feasible fractional solution}}$$

For each i, S

$$\underbrace{v_i(S_i^*) - \sum_{j \in S_i^*} p_j^*}_{u_i(S_i^*)} \geq \underbrace{v_i(S) - \sum_{j \in S} p_j^*}_{v_i(S)}$$

$u_i(S_i^*) \geq u_i(S)\ \forall S$, in every feasible solution $\forall i\ \sum_S X^*_{i,S} \leq 1$

$$v_i(S) - \sum_{j \in S} \sum p^*_j \geq \sum_S X^*_{i,S}[v_i(S) - \sum_{j \in S} p^*_j]$$

$$\Rightarrow v_i(S^*_i) - \sum_{j \in S^*_i} p^*_j \geq \sum_S X_{i,S}[v_i(S) - \sum_{j \in S} p^*_j] = \sum_S X^*_{i,S} v_i(S) - \sum_S X^*_{i,S} p^*_j$$

Now, sum over all players $i \in N$:

$$\sum_{i \in N} v_i(S^*_i) - \sum_{i \in N} \sum_{j \in S^*_i} p^*_j \geq \sum_{i \in N, S \subseteq M} X^*_{i,S} v_i(S) - \sum_{i \in N, S \subseteq M} X^*_{i,S} p^*_j$$

It remains to show that

$$\sum_{i \in N, j \in S} p^*_i \geq \sum_{i \in N, S \subseteq M} X^*_{i,S} p^*_j$$

We get this by $\sum_{i \in N, j \in S} p^*_i = \sum_{j \in M} p^*_j$ and $\sum_{i \in N, S \subseteq M} X^*_{i,S} p^*_j \leq \sum_{j \in M} p^*_j$.

■

Theorem 6 [Second Welfare Theorem] *If there exists an optimal integral solution to a Cofiguration LPR then there exists a* Walrasian Equilibrium *and it gives us the allocation for the solution.*

From Theorem 6 the following characterization is implied:

Characterization 7 *A* Walrasian Equilibrium *exists* $\iff$ *LPR has an optimal, integral solution.*

Example 8 *We consider an example of a market with 2 players and 2 products. The players have the following valuations:*

$$v_1(a) = v_1(b) = v_1(a, b) = 2$$

$$v_2(a, b) = 3$$

from The First Welfare Theorem we get that in every WE, $S_1 = \emptyset$ *and* $S_2 = \{a, b\}$*. We get* $p_a + p_b \leq 3 \Rightarrow \exists_{i \in \{a,b\}} p_i \leq 1.5$ *but in this case player 1 will want product i for this price.*

The optimal integral solution:

	a	b
1	2	2
2	3	

A fractional solution that satisfies all the constraints:

$$X_{1,a} = \frac{1}{2}$$

$$X_{1,b} = \frac{1}{2}$$

$$X_{2,\{a,b\}} = \frac{1}{2}$$

$\frac{1}{2}v_1(a) + \frac{1}{2}v_1(b) + \frac{1}{2}v_2(\{a,b\}) = 1 + 1 + 1\frac{1}{2} = 3\frac{1}{2}$.

We got $3\frac{1}{2}$ *while in the optimal integral solution what we got was* 3*.*

4 Gross Substitution Valuations Class

The *Gross Substitution* (GS) valuations class is a subset of the *Sub Modular* (SM) valuation class, that we have encountered thus far.

$$GS \subseteq SM$$

A player's valuation is GS: price vector $\overrightarrow{p} = (p_1, ..., p_n)$; a demand vector, e.g. $D_i = \{1, 2, 4, 17\}$; a non decreasing prices vector $\overrightarrow{q} = (\ddot{q}_1, \ddot{q}_2, \ddot{q}_3, \hat{q}_4, ...)$.

Theorem 9 *In every environment where all the players have a GS valuation, there always exists a WE.*

The proof for Theorem 9, as well as more details on the subject can be found in Chapter 11 of the book *Algorithmic Game Theory, Cambridge University Press 2007.*