Lecture 11

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1 Stable Matching

In the matching model we are given two sets, M ("Men") and W ("Women") of the same size $|M| = |W| = n$. Each man in M ranks all women in W in order of preference. Similarly women rank men.

Definition 1 *A **Stable Matching** is a perfect matching, i.e., every member of one group is matched to exactly one member of the other group, such that for every pair (m, w) that is not matched at least one of these conditions holds:*

- *m is matched to a woman w' that is ranked higher than w*
- *w is matched to a man m' that is ranked higher than m*

If neither of the two conditions hold, the pair (m, w) is called a ***blocking pair***.

1.1 The Gale-Shapley algorithm [1]

We present here a version of the algorithm where the men propose to the women. A version where the women propose to the men is analogous.

Algorithm 1 Find a stable matching

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while there is a man m without a match do
    m proposes to the best woman that did not reject him already
    Every woman holds her best offer
end while
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An example of an instance of the problem and a matching produced by the algorithm is given in Figure 1.

Observation 2 *Throughout the algorithm, every woman can only improve her state.*

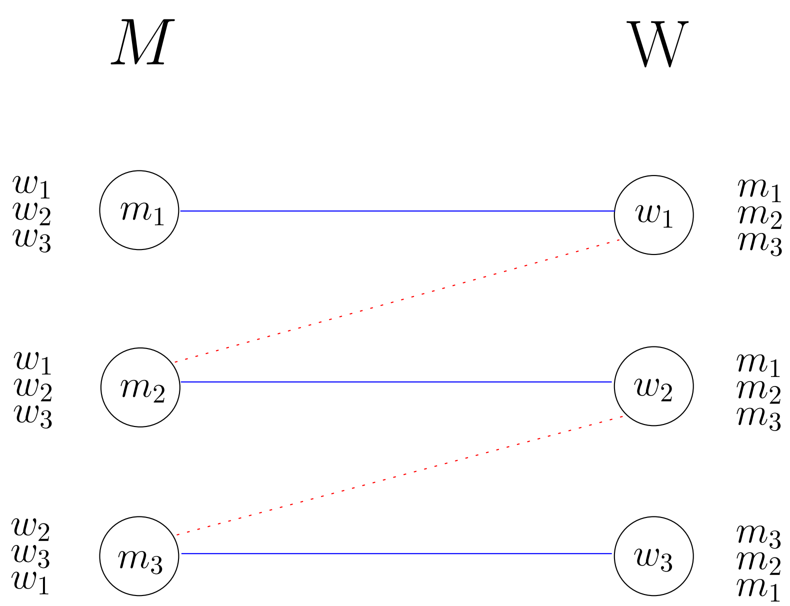

Figure 1: Example of running the Gale-Shapley algorithm. Blue edges represent the matches and the red edges rejections.

Observation 3 *Throughout the algorithm, every woman is matched to at most one man. The same applies to the men.*

Theorem 4 *The algorithm finds a stable matching in* $O(n^2)$ *time.*

Proof: Every man proposes to at most n women and therefore there are at most n^2 iterations.

We now show that the algorithm ends in a perfect matching. Assume by contradiction that the algorithm does not end in perfect matching. Therefore there is a man that is not matched, i.e., he was rejected by **all** women. A man is rejected by a woman only in favor of a better man for her. When a woman is matched to someone, she remains matched, perhaps to someone else, until the algorithm terminates. Hence all the women are matched at the end of the algorithm and therefore so are the men.

We finally show the matching is stable. Let (m, w) be a pair that is not matched. This can happen in one of two cases:

Case a: m never proposed to w.

So m is matched with w' which ranks higher than w on his list.

Case b: m proposed to w.

Since they ate not matched, w rejected him for someone better. Therefore, when the algorithm terminates, w will be matched to m' she prefers over m.

■

Notice that a stable matching is not unique. The group that proposes affects the chosen matching as presented in Figure 2.

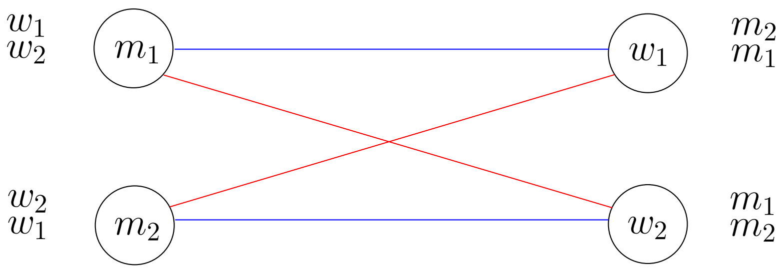

Figure 2: Example of effect of the sides proposing during the algorithm. When the men propose we get the blue matching and when the women propose we get the red matching.

Let $S(m)$ be a set of all women that m is matched to in some stable matching. Denote by $Best(m)$ the highest ranked woman $w \in S(m)$.

Theorem 5 *When the men choose the women, the algorithm outputs a matching where every man m is matched to $Best(m)$.*

Proof: Let $R = (m, w)$ be a set of all pairs such that w rejected m at some point during the algorithm.

Since every man only decreases his preferences as the algorithm progresses, if m is matched to w at the end of the algorithm then every other woman w' that m prefers over w would have already rejected him. Therefore it is enough to show that for every pair $(m, w) \in R$, there is no stable matching that matches (m, w).

We will prove the claim by induction on the number of iterations of the algorithm.

Initially $R = \emptyset$.

Lets look on the iteration where w rejects m for m'. So either m or m' proposed to w in this iteration.

Since m' is "serial" (goes over his preferences from high to low), for every w' he prefers over w, $(m', w') \in R$. By the induction hypothesis, no stable matching matches m' to a woman he prefers over w. Therefore, in every stable matching m' is matched with someone he prefers less than w.

Since w prefers m' over m, and m' prefers w over every other woman he could be matched in a stable matching, there is no stable matching that matched m and w (otherwise (m', w) would have been a blocking pair.

In other words, on the one hand, if w rejected m for m' then w prefers m' over m. On the other hand, m' prefers w over every other woman he could have been be matched in a stable matching. ■

2 Revenue Maximization

Simple model:
One product, one buyer. v - the value of the product. *Posted price* (truthfull mechanism) - the seller publishes a price r, and if $v \geq r$ the seller gets *revenue* of r else the revenue is 0 .

Bayesian model:
A player i has a private value v_i where v_i is sampled from a distribution F_i with a density function f_i on $[0, v_{max}]$.

$$F_i(z) = P_r(v_i \leq z)$$

We assume that the distributions $F_1, \ldots, F_n$ are independent and known.

The target function:
Maximizing expected profit $E[Rev]$.

$$E[Rev] = r(1 - F(r))$$

Example 1: If

$$v_i \sim u[0, 1]$$

$$F(r) = r$$

then

$$E[Rev] = r(1 - r)$$

By deriving and comparing to zero we get that $r = 1/2$ maximizes $E[Rev]$.

Example 2: Slightly different model. One product, two buyers each with a value of $v_i \sim u[0, 1]$

In a second price auction

$$E[Rev] = E[V_2] = 1/3$$

We define a *second price auction with minimum price* r as second price auction where a player must bid at least r in order to win. For example, if choose $r = 1/2$ as a minimum price

- with Probability of $1/4$: $v_1 < 1/2$ and $v_2 < 1/2 \rightarrow rev = 0$
- with Probability of $1/4$: $v_1 > 1/2$ and $v_2 > 1/2 \rightarrow rev = 2/3$
- with Probability of $1/2$: $(v_1 > 1/2$ and $v_2 < 1/2)$ or $(v_1 > 1/2$ and $v_2 < 1/2) \rightarrow rev = 1/3$

and we get

$$E[Rev] = 1/4 * 0 + 1/4 * 2/3 + 1/2 * 1/2 = 5/12 > 1/3$$

Reminder:

- According to the Revelation Principle, $\forall i, b_i = v_i$
- An auction mechanism is a pair (x, p) s.t. x is the allocation rule and p is the payment rule.
- Revenue expectation is $\mathbb{E}_{v \sim F}[\sum_{i=1}^{n} P_i(v)]$

According to Mayerson's lemma [2], the payments for a monotone allocation are:

$$P_i(b_i, b_{-i}) = \int_0^{b_i} z x_i'(z, b_{-i}) \, \mathrm{d}z \tag{1}$$

for a player i and v_{-i}. And the expected revenue is

$$\mathbb{E}_{v_i \sim F_i}[P_i(v)] = \int_0^{V_{max}} P_i(v) f_i(v_i) \, \mathrm{d}v_i \tag{2}$$

Combining equations 1 and 2 we get

$$= \int_0^{V_{max}} \left(\int_0^{b_i} z x_i'(z, b_{-i}) \, \mathrm{d}z] \right) f_i(v_i) \, \mathrm{d}v_i$$

Replacing the integration order gives

$$= \int_0^{V_{max}} \left(\int_z^{V_{max}} f_i(v_i) \, \mathrm{d}v_i \right) z x_i'(z, b_{-i}) \, \mathrm{d}z$$

We can see that

$$\int_z^{V_{max}} f_i(v_i) \, \mathrm{d}v_i = 1 - F_i(z)$$

and hence the expected revenue is

$$= \int_0^{V_{max}} (1 - F_i(z)) z x_i'(z, b_{-i}) \, \mathrm{d}z$$

We use partial integration with

$$f(z) \equiv (1 - F_i(z))z, g'(z) \equiv x_i'(z, v_{-i})$$

and get

$$= (1 - F_i(z)) z x_i(z, v_{-i})|_0^{V_{max}} - \int_0^{V_{max}} (1 - F_i(z) - z f_i(z)) x_i(z, v_{-i}) \mathrm{d}z$$

Since $F_i(z) = 1$ we get

$$= - \int_0^{V_{max}} (1 - F_i(z) - z f_i(z)) x_i(z, v_{-i}) \mathrm{d}z = \int_0^{V_{max}} \left(z - \frac{1 - F_i(z)}{f_i(z)} \right) x_i(z, v_{-i}) f_i(z) \mathrm{d}z$$

The expression we got is similar to expectancy of the "virtual value of z" defined as

$$\phi(z) = z - \frac{1 - F_i(z)}{f_i(z)} \rightarrow \int_0^{V_{max}} \phi(z) x_i(z, v_{-i}) f_i(z) \mathrm{d}z$$

so you get

$$\mathbb{E}_{v_i \sim F_i}[P_i(v)] = E_{v_i \sim F_i}[\phi_i(v_i) x_i(v)]$$

and if we sum over all players and use the linearity of the expectation we get

$$\underbrace{E_v[\sum_{i=1}^{n} P_i(v)]}_{\textit{expectation on the revenue}} = \underbrace{E_v[\sum_{i=1}^{n} \phi_i(v_i) x_i(v)]}_{\textit{expectation on the virtual fare}} \tag{3}$$

One product auction:

According to Equation 3, in order to maximize $E[Rev]$ we need to maximize

$$E_v[\sum \theta_i(v_i) x_i(v)]$$

under the constraint

$$\sum x_i(v) \leq 1$$

where the minimum price is

$$\theta_i^{-1}(0)$$

In order to maximize $E[Rev]$ we give the product to the player that maximizes $\theta_i(v_i)$ as long as it is positive. Otherwise, no player gets the product.

Definition 6 *A distribution F called **regular** if the function $\theta_i(v_i) = v_i - 1 - F_i(v_i)/f_i(v_i)$ is increasing in v_i.*

For example, the uniform and exponential distributions are regular.

Conclusion:

Second price auction with minimum price of $\theta_i^{-1}(0)$ is optimal over all the truthful mechanisms

Definition 7 *A strategy profile* $(b_1(v_1), ..., b_n(v_n))$ *is in* ***Bayesian equilibrium*** *if* $\forall i, v_i$ $b_i(v_i)$ *is the best action for* $b_{-i}(v_{-i})$ *where* $v_{-i} \sim F_{-i}$.

References

[1] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[2] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.