Bayesian price of anarchy
Simultaneous first and second price auctions

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Speaker: Ophir Friedler

Combinatorial Auctions

- m items $M = \{1 \ldots m\}$
- n bidders $N = \{1 \ldots n\}$
- Valuation function per bidder i: $v_i : 2^M \to \mathbb{R}$

Two Assumptions

- Normalization: $v_i(\phi) = 0$
- Monotonicity: $\forall S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$

Objective: Find a partition of the items $X = X_1 \ldots X_n$ that maximizes social welfare:

$$SW(X) = \sum_{i \in N} v_i(X_i)$$

Simultaneous Auctions

- m items are sold in m independent (simultaneous) first/second price auctions
- Each bidder i submits an m-vector of bids $b_i(j)$
- Define $W_i(\mathbf{b})$ to be the set of items won by bidder i.
- Define $p_i(S)$ to be the sum of payments bidder i needs to pay for the set S.
- Each bidder i has a quasi-linear utility:

$$u_i(\mathbf{b}) = v_i(W_i(\mathbf{b})) - p_i(W_i(\mathbf{b}))$$

Example (bad)

Consider 2 bidders and 1 item.

$$v_1(1) = 1 \qquad\qquad v_2(1) = 0$$

- Optimal social welfare is when bidder 1 gets item (SW $= 1$)
- Bad PNE: bidder 1 bids 0, bidder 2 bids 1 (SW $= 0$).

Conclusion: The PoA may be infinity in the general case

Overbidding

Two versions of overbidding:

- A bid vector **b** satisfies *weak* no overbidding if:

$$\sum_{j \in W_i(\mathbf{b})} b_{ij} \leq v_i(W_i(\mathbf{b}))$$

- A bid vector **b** satisfies *strong* no overbidding if:

$$\forall S \subseteq M : \sum_{j \in S} b_{ij} \leq v_i(S)$$

Example (PoA $= \frac{1}{2}$)

Consider 2 bidders and 2 items $\{x, y\}$ and the following valuations:

$$v_1(\{x\}) = v_1(\{x,y\}) = 2 \qquad v_2(\{y\}) = v_2(\{x,y\}) = 2$$
$$v_1(\{y\}) = 1 \qquad v_2(\{x\}) = 1$$

A PNE with optimal SW (of 4) exists, however the bid profile:

$$b_{1x} = 0, b_{1y} = 1, b_{2x} = 1, b_{2y} = 0$$

Satisfies *strong* no overbidding, it is a PNE, and has $SW = 2$.
(Each bidder gets the item that he wants less)

Types of valuations

Submodular

A valuation v_i is *submodular* if for all $S, T \subseteq M$:
$v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$
Equivalently (decreasing marginal utilities): for all $S \subseteq T$:
$v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T)$

XOS (a.k.a fractionally subadditive)

A valuation v_i is *fractionally-subadditive* if there are additive valuations $A = \{a_1 \ldots a_l\}$ where for each a_r and set S:
$a_i(S) = \sum_{j \in S} a_i(\{j\})$ and for all $S \subseteq M$ it holds that
$v_i(S) = \max_{a \in A} a(S)$

Subadditive

A valuation is subadditive if for all
$S, T \subseteq M : v_i(S) + v_i(T) \geq v_i(S \cup T)$

submodular $\subset$ XOS $\subset$ subadditive

XOS example

Consider 3 items - Apple , Tomato,and Fork.

The valuation v is XOS with $A = \{a_1, a_2\}$:

	Apple	Tomato	fork
a_1	0	2	1
a_2	3	1	0

$v(\text{Apple}, \text{Tomato}) = \max\{0 + 2, 3 + 1\} = 4$
$v(\text{Tomato}, \text{Fork}) = \max\{2 + 1, 1 + 0\} = 3$

PoA Bounds

Theorem (unproven)

If players are strongly not overbidding, for every subadditive valuation profile, every PNE has welfare at least $\frac{1}{2}$ of optimal — i.e., the POA is at most 2

Problems

- A pure NE rarely exists
- The full-information assumption (valuations are common knowledge) is often unrealistic

Bayesian Setting (model of incomplete information)

- There is a set of possible valuations V_i for bidder i
- There is a known distribution F_i over V_i
- $F = \times_i F_i$ the common prior distribution (C.P.D)
- Each bidder i knows its own private v_i
- A *strategy* s_i maps valuations (a.k.a types) to (possibly randomized) actions (bid vectors)

Definition: A strategy profile s is a Bayes-Nash Equilibrium (BNE) if for every bidder i, type v_i, and any other bidding strategy $\tilde{s}$ it holds:

$$\mathop{\mathbb{E}}_{v_{-i} \sim F_{-i}} \left[\mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})} [u_i(\mathbf{b})] \right] \geq \mathop{\mathbb{E}}_{v_{-i} \sim F_{-i}} \left[\mathop{\mathbb{E}}_{\substack{\mathbf{b_{-i}} \sim \mathbf{s}(\mathbf{v_{-i}}) \\ \tilde{b}_i \sim \tilde{s}_i}} [u_i(\tilde{b}_i, \mathbf{b_{-i}})] \right]$$

Bayesian Setting (model of incomplete information)

Given a valuation profile v, denote by $OPT^v = (OPT_1^v \dots OPT_n^v)$ the welfare-maximizing assignment for profile v.

Bayesian Price of Anarchy

To measure the optimal social welfare in a game of incomplete information we take the expected optimal welfare:

$$BPoA = \sup_{\substack{(F,s) \\ s \text{ is BNE for } F}} \frac{\mathbb{E}_{v\sim F}\left[\text{optimal welfare for } v\right]}{\mathbb{E}_{v\sim F}\left[\text{welfare of } s(v)\right]}$$

$$= \sup_{\substack{(F,s) \\ s \text{ is BNE for } F}} \frac{\mathbb{E}_{v\sim F}\left[\sum_{i\in N} v_i(OPT_i^v)\right]}{\mathbb{E}_{v\sim F, b\sim s(v)}\left[\sum_{i\in N} v_i(W_i(\mathbf{b}))\right]}$$

Where $W_i(\mathbf{b})$ denotes the set of items i wins given bid vector $\mathbf{b}$

Fractionally subadditive valuations

Theorem 1

Let F be a C.P.D over fractionally-subadditive valuations of bidders. If **s** *is a BNE of a simultaneous second-price auction, and the bidders are weakly no-overbidding then:*

$$\frac{\mathbb{E}_{v\sim F}\left[\sum_{i\in N} v_i(OPT_i^v)\right]}{\mathbb{E}_{v\sim F}\left[\sum_{i\in N} v_i(W_i(s(v)))\right]} \leq 2$$

Note that for simplicity we consider only pure bids (i.e. $s(v)$ is a vector of bids, and not a distribution over bid vectors).

Key lemma

The following lemma shows how to exploit the property of fractionally-subadditive valuations:

Lemma 1

Let a_i be a maximizing additive valuation of bidder i for set S, and $a_i(j) = 0$ for all $j \in M \setminus S$, then:

$$u_i(a_i, b_{-i}) \geq v_i(S) - \sum_{j \in S} \max_{k \neq i} b_k(j)$$

Proof of lemma 1

- Bidder i gets $W_i := W_i(a_i, b_{-i})$
- WLOG $W_i \subseteq S$ (if i gets some $j \notin S$ then the maximum bid on it was 0)
- For every non-obtained item $j \in S \setminus W_i$ it holds $a_i(j) - \max_{k \neq i} b_k(j) \leq 0$

Therefore:

$$
\begin{aligned}
u_i(a_i, b_{-i}) &= v_i(W_i) - \sum_{j \in W_i} \max_{k \neq i} b_k(j) \\
&\geq \sum_{j \in W_i} a_i(j) - \sum_{j \in W_i} \max_{k \neq i} b_k(j) && \text{(XOS)} \\
&\geq \sum_{j \in S} a_i(j) - \sum_{j \in S} \max_{k \neq i} b_k(j) && (S \setminus W_i \text{ items}) \\
&= v_i(S) - \sum_{j \in S} \max_{k \neq i} b_k(j) && \text{(XOS)}
\end{aligned}
$$

Proof of theorem 1

- Fix a valuation profile $v = (v_1 \dots v_n)$, and $OPT^v = (OPT_1^v \dots OPT_n^v)$ the optimal allocation with respect to v.
- a_i is the maximizing additive valuation for OPT_i^v
- Consider a BNE profile s:

$$\mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}} \left[u_i(s_i(v_i), s_{-i}(w_{-i})) \right] \geq \mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}} \left[u_i(a_i, s_{-i}(w_{-i})) \right]$$

- $W_i^{w_{-i}}$ is the set i wins given s and the valuation profile w_{-i}:

$$W_i^{w_{-i}} = W_i(s_i(v_i), s_{-i}(w_{-i}))$$

Proof of theorem 1

- $v_i(W_i^{w_{-i}}) \geq u_i(s_i(v_i), s_{-i}(w_{-i}))$ by dropping the payments.
- Applying Lemma 1:

$$u_i(a_i, s_{-i}(w_{-i})) \geq v_i(OPT_i^v) - \sum_{j \in OPT_i^v} \max_{k \neq i} s(w_k)(j)$$

- Taking expectation:

$$\mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}}\left[v_i(W_i^{w_{-i}})\right] \geq \mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}}\left[v_i(OPT_i^v) - \sum_{j \in OPT_i^v} \max_{k \neq i} s(w_k)(j)\right]$$

$$= v_i(OPT_i^v) - \mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}}\left[\sum_{j \in OPT_i^v} \max_{k \neq i} s(w_k)(j)\right]$$

Proof of theorem 1

- Another expectation on $w_i \sim F_i$:

$$= v_i(OPT_i^v) - \mathop{\mathbb{E}}_{w \sim F}\left[\sum_{j \in OPT_i^v} \max_{k \neq i} s(w_k)(j)\right]$$

- Taking a maximum over a larger set:

$$\geq v_i(OPT_i^v) - \mathop{\mathbb{E}}_{w \sim F}\left[\sum_{j \in OPT_i^v} \max_{k \in N} s(w_k)(j)\right]$$

Proof of theorem 1

- We obtained:

$$v_i(OPT_i^v) \le \mathop{\mathbb{E}}_{w \sim F}\left[\sum_{j \in OPT_i^v} \max_{k \in N} s(w_k)(j)\right] + \mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}}\left[v_i(W_i^{w_{-i}})\right]$$

- By summing over all i and and taking expectation over all v, we can bound the expected optimal social welfare:

$$\mathop{\mathbb{E}}_{v \sim F}\left[\sum_{i \in N} v_i(OPT_i^v)\right] \le \mathop{\mathbb{E}}_{v \sim F}\left[\sum_{i \in N} \mathop{\mathbb{E}}_{w \sim F}\left[\sum_{j \in OPT_i^v} \max_{k \in N} s(w_k)(j)\right]\right] + \mathop{\mathbb{E}}_{v \sim F}\left[\sum_{i \in N} \mathop{\mathbb{E}}_{w_{-i} \sim F_{-i}}\left[v_i(W_i^{w_{-i}})\right]\right]$$

Proof of theorem 1

Lets bound each of the summands, first summand:

$$\mathop{\mathbb{E}}_{v\sim F}\left[\sum_{i\in N}\mathop{\mathbb{E}}_{w\sim F}\left[\sum_{j\in OPT_i^v}\max_{k\in N} s(w_k)(j)\right]\right] = \qquad \text{(L.O.E)}$$

$$\mathop{\mathbb{E}}_{v\sim F}\left[\mathop{\mathbb{E}}_{w\sim F}\left[\sum_{i\in N}\sum_{j\in OPT_i^v}\max_{k\in N} s(w_k)(j)\right]\right] =$$

(OPT^v is a partition, sum over items and v disappears)

$$\mathop{\mathbb{E}}_{w\sim F}\left[\sum_{j\in M}\max_{k\in N} s(w_k)(j)\right] = \mathop{\mathbb{E}}_{w\sim F}\left[\sum_{i\in N}\sum_{j\in W_i(s(w))} s(w_i)(j)\right] \le$$

(no overbidding)

$$\mathop{\mathbb{E}}_{w\sim F}\left[\sum_{i\in N} v_i(W_i(s(w)))\right]$$

This is the expected welfare from the BNE profile s

Proof of theorem 1

Second summand:

$$\mathop{\mathbb{E}}_{v\sim F}\left[\sum_{i\in N}\mathop{\mathbb{E}}_{w_{-i}\sim F_{-i}}\left[v_i(W_i^{w_{-i}})\right]\right]= \qquad \text{(L.O.E)}$$

$$\sum_{i\in N}\mathop{\mathbb{E}}_{v\sim F}\left[\mathop{\mathbb{E}}_{w_{-i}\sim F_{-i}}\left[v_i(W_i^{w_{-i}})\right]\right]= \qquad \text{(independence)}$$

$$\sum_{i\in N}\mathop{\mathbb{E}}_{v_i\sim F_i}\left[\mathop{\mathbb{E}}_{w_{-i}\sim F_{-i}}\left[v_i(W_i^{w_{-i}})\right]\right]= \qquad \text{(renaming)}$$

$$\sum_{i\in N}\mathop{\mathbb{E}}_{v\sim F}\left[v_i(W_i^{v_{-i}})\right]=\mathop{\mathbb{E}}_{v\sim F}\left[\sum_{i\in N}v_i(W_i(s(v)))\right]$$

This is the expected welfare from the BNE profile s

Proof of theorem 1

- Altogether we get

$$\mathop{\mathbb{E}}_{v\sim F}\left[\sum_{i\in N} v_i(OPT_i^v)\right] \leq 2 \cdot \mathop{\mathbb{E}}_{v\sim F}\left[\sum_{i\in N} v_i(W_i(s(v)))\right]$$

Which completes the theorem.

Going beyond fractionally sub-additive

Q. Can we hope for good efficiency bounds for the general BNE?

A. For general valuations - the answer is no. Intuitively consider a buyer with a large value for the set of all objects, but no value for anything less. Any positive bid carries a great risk of winning some but not all items.

- (without proof): The mixed PoA is as high as $\Omega(\sqrt{m})$ for first- and second-price auctions.

Going beyond fractionally sub-additive

The good news:

The presence of complements is the only barrier to a constant Bayesian PoA

Theorem 2

When buyers have subadditive valuations, the BPoA for simultaneous first-price item auctions is at most 2.

Theorem 3

When buyers have subadditive valuations, the BPoA for simultaneous second-price item auctions is at most 4, under the strong/weak no-overbidding assumption

How to approach the problem

The proof for the theorems is divided into 2 parts

Part 1

Consider a single player game: bidding against a distribution over price vectors, and find a good strategy for that player.

Part 2

In the original game, each player will use the strategy from part 1 on an "optimal" subset of items, where the distribution of prices is induced by the game-play of other players.

A vector will be considered as an additive function:
Given an m-vector p and a set $S \subseteq [m]$: $p(S) = \sum_{j \in S} p_j$

Part 1

As with the case of XOS valuation functions, the essential arguments will need to exploit the sub-additive property.

Consider the following setup:

- Single bidder
- m items
- Price distribution D
- $W(b, p)$ is the set of items the bidder wins when bidding b at price p: $W(b, p) = \{j \in [m] | b(j) \geq p(j)\}$
- $v(b, p) = v(W(b, p))$ for shorter writing

We are looking for a bidding strategy b such that:

$$\mathop{\mathbb{E}}_{p \sim D}[v(b, p)] - b([m]) \geq \alpha v([m]) - \mathop{\mathbb{E}}_{p \sim D}[p([m])] \qquad (1)$$

Part 1 - finding a good bidding strategy

For a vector of fixed prices p, the problem is trivial:

- Bid $b = p$:

$$\mathop{\mathbb{E}}_{p\sim D}[v(b,p)] - b([m]) = \mathop{\mathbb{E}}_{p\sim D}[v([m])] - b([m]) = v([m]) - p([m])$$

Bidding against random prices is harder, but:

Lemma 2

For any distribution D of prices p and any sub-additive valuation $v(\cdot)$ there exists a bid b_0 such that

$$\mathop{\mathbb{E}}_{p\sim D}[v(b_0,p)] - b_0([m]) \geq \frac{1}{2}v([m]) - \mathop{\mathbb{E}}_{p\sim D}[p([m])]$$

Proof of lemma 2

The method:

- Show the lemma is true in expectation.
- Infer the existence of such a bid.

By renaming the random variable:

$$\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v(b,p)\right]\right] = \mathop{\mathbb{E}}_{p\sim D}\left[\mathop{\mathbb{E}}_{b\sim D}\left[v(b,p)\right]\right]$$

Therefore:

$$= \frac{1}{2}\cdot\left(\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v(b,p)\right]\right] + \mathop{\mathbb{E}}_{p\sim D}\left[\mathop{\mathbb{E}}_{b\sim D}\left[v(b,p)\right]\right]\right)$$

By renaming the random variables and linearity of expectation:

$$\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v(b,p)\right]\right] = \frac{1}{2}\cdot\left(\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v(b,p)+v(p,b)\right]\right]\right)$$

Proof of lemma 2

Plug in sub-additivity of $v(\cdot)$:

$$v(b,p) + v(p,b) \geq v([m])$$

Therefore:

$$\frac{1}{2}\cdot\left(\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v(b,p)+v(p,b)\right]\right]\right) \geq \frac{1}{2}\cdot\left(\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v([m])\right]\right]\right) = \frac{1}{2}v([m])$$

Add the bids to the equation:

$$\begin{aligned}\mathop{\mathbb{E}}_{b\sim D}\left[\mathop{\mathbb{E}}_{p\sim D}\left[v(b,p) - b([m])\right]\right] &\geq \frac{1}{2}v([m]) - \mathop{\mathbb{E}}_{b\sim D}\left[b([m])\right] \\ &= \frac{1}{2}v([m]) - \mathop{\mathbb{E}}_{p\sim D}\left[p([m])\right]\end{aligned}$$

Since a bid drawn from D satisfies the inequality in expectation, a bid b_0 that satisfies the lemma must exist.

The little issue of no-overbidding

- The 'good bid' from Lemma 2 is not guaranteed to satisfy any no-overbidding assumption.
- Lemma (without proof): For a given price vector p and any sub-additive valuation $v(\cdot)$ there exists a strong no-overbidding bid b such that $v(b, p) - b([m]) \geq v([m]) - p([m])$

Lemma 3

For any distribution of prices D and any sub-additive valuation $v(\cdot)$ there exists a strong no-overbidding bid b_0 such that:

$$\mathop{\mathbb{E}}_{p \sim D}[v(b_0, p)] - b_0([m]) \geq \frac{1}{2} v([m]) - \mathop{\mathbb{E}}_{p \sim D}[p([m])]$$

Proof of lemma 3

- Fix some q from the support of D
- Let $T \subseteq [m]$ be a maximal set such that $v(T) \leq q(T)$
- $\tilde{q}$ is the *truncated* such that:
  $\forall j \in T : \tilde{q}_j = 0,\ \forall j \in [m] \setminus T : q_j = \tilde{q}_j$
- Observation:$\forall R \subseteq [m] \setminus T\ :\ v(R) > q(R)$
  Otherwise:

  $$v(R \cup T) \leq v(R) + V(T) \leq q(R) + q(T) = q(R \cup T)$$

  Which contradicts the maximality of T
- $\tilde{q}$ is a strongly no-overbidding bid vector:

  $$\forall S \subseteq [m] : \tilde{q}(S) \leq \tilde{q}((S \setminus T) \cup T) = \\ \tilde{q}(S \setminus T) < v(S \setminus T) \leq v(S)$$

Proof of lemma 3

- $\tilde{D} = \{\tilde{q} | q \sim D\}$ is the distribution of truncated prices.
- Assume: (will be shown)

$$v(b, q) + q([m]) \geq v(b, \tilde{q}) + \tilde{q}([m])$$

- Fix b, take expectation on $p \sim D$:

$$\mathop{\mathbb{E}}_{p \sim D}[v(b, p) + p([m])] \geq \mathop{\mathbb{E}}_{\tilde{p} \sim \tilde{D}}[v(b, \tilde{p}) + \tilde{p}([m])]$$

- Take expectation for $b \sim \tilde{D}$:

$$\mathop{\mathbb{E}}_{b \sim \tilde{D}}\left[\mathop{\mathbb{E}}_{p \sim D}[v(b, p) + p([m])]\right] \geq \mathop{\mathbb{E}}_{b \sim \tilde{D}}\left[\mathop{\mathbb{E}}_{\tilde{p} \sim \tilde{D}}[v(b, \tilde{p}) + \tilde{p}([m])]\right]$$

$$= \mathop{\mathbb{E}}_{b \sim \tilde{D}}\left[\mathop{\mathbb{E}}_{\tilde{p} \sim \tilde{D}}[v(b, \tilde{p})]\right] + \mathop{\mathbb{E}}_{b \sim \tilde{D}}[b([m])]$$

- In the same manner in proof of lemma 2:

$$\geq \frac{1}{2} v([m]) + \mathop{\mathbb{E}}_{b \sim \tilde{D}}[b([m])]$$

Proof of lemma 3

Reorganizing the inequality we get:

$$\mathbb{E}_{b\sim\tilde{D}}\left[\mathbb{E}_{p\sim D}\left[v(b,p) - b([m])\right]\right] \geq \mathbb{E}_{b\sim\tilde{D}}\left[\frac{1}{2}v([m]) - \mathbb{E}_{p\sim D}\left[p([m])\right]\right]$$

Which proves Lemma 3 in expectation over strongly no-overbidding bids.
At least one bid vector must satisfy

Proof of lemma 3

Let us now show that:

$$v(b, q) + q([m]) \geq v(b, \tilde{q}) + \tilde{q}([m])$$

- $W(b, \tilde{q}) \subseteq W(b, q) \cup T$: bids are 0 on T, identical on the rest
- By sub-additivity: $v(b, \tilde{q}) \leq v(b, q) + v(T)$
- $q([m]) - \tilde{q}([m]) = q(T) \geq v(T)$
- Altogether:

$$v(b, \tilde{q}) \leq v(b, q) + v(T) \leq v(b, q) + q([m]) - \tilde{q}([m])$$

Which proves the inequality

Mid-way Summary

So what have we achieved so far?

- We've shown existence of a good bidding strategy b_0 against a distribution D over price vectors that achieves the inequality:

$$\mathop{\mathbb{E}}_{p \sim D}[v(b_0, p)] - b_0([m]) \geq \frac{1}{2} v([m]) - \mathop{\mathbb{E}}_{p \sim D}[p([m])]$$

- We have shown that such a bid exists even if we want it to be strongly no-overbidding with regard to some valuation function $v(\cdot)$

Part 2 - Returning to simultaneous first-price auctions

Given some C.P.D F, we want to bound the BPoA, the method:

- Fix some BNE s
- Draw a virtual type profile $v^*_{-i} \sim F_{-i}$, denote $v^* = (v_i, v^*_{-i})$
- Consider an optimal allocation OPT^{v^*}
- Consider a price distribution on the set $OPT_i^{v^*}$ by drawing $v_{-i} \sim F$
- Bid 'good' on the set $OPT_i^{v^*}$
- Exploit the equilibrium property to get a bound on BPoA

Proof of theorem 2

The setup:

- Fix a type distribution $F = \times_{i=1}^{n} F_i$
- Let s be a BNE for F
- Fix i and v_i
- Fix an arbitrary v_{-i}, denote $v = (v_i, v_{-i})$
- Fix an arbitrary v^*_{-i}, denote $v^* = (v_i, v^*_{-i})$
- $b \sim s(v)$ are the randomized bids given v
- $\varphi_i(b_{-i})$ is the price vector b_{-i} induces on i
- Let p be equal to $\varphi_i(b_{-i})$ on OPT^{v^*} and 0 elsewhere
- D is the distribution over maximum bids in b_{-i} on items in $OPT_i^{v^*}$.
  I.e. D is the distribution over p where $b \sim s(v)$.

Proof of theorem 2

Replace $[m]$ by $OPT_i^{v^*}$ in lemma 2, let b_i' be the bid vector s.t.:

$$\mathop{\mathbb{E}}_{p\sim D}\left[v_i(b_i',p)\right] - b_i'(OPT_i^{v^*}) \geq \frac{1}{2}v_i(OPT_i^{v^*}) - \mathop{\mathbb{E}}_{p\sim D}[p(OPT_i^{v^*})]$$

Since s is a BNE:

$$\mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}\left[u_i(b)\right] \geq \mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}\left[u_i(b_i', b_{-i})\right]$$

By definition of quasi-linear utilities:

$$= \mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}\left[v_i(b_i', \varphi_i(b_{-i}))\right] - \mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}\left[b_i'(W_i(b_i', b_{-i}))\right]$$

$$= \mathop{\mathbb{E}}_{p\sim D}\left[v_i(b_i',p)\right] - \mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}\left[b_i'(W_i(b_i', b_{-i}))\right]$$

(In first-price the payment on an item you won is your bid)

Proof of theorem 2

- $W_i(b', b_{-i}) \subseteq OPT^{v^*}$ for all b_{-i} therefore:

$$\mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}\left[u_i(b)\right] \geq \mathop{\mathbb{E}}_{p\sim D}\left[v_i(b'_i, p)\right] - b'_i(OPT^{v^*})$$

- Plug in the guarantee on b_i:

$$\mathop{\mathbb{E}}_{v_{-i}\sim F_{-i}, b\sim s(V)}\left[u_i(b)\right] \geq \frac{1}{2} v_i(OPT_i^{v^*}) - \mathop{\mathbb{E}}_{p\sim D}[p(OPT_i^{v^*})]$$

Proof of theorem 2

Re-stating according to the definiton of D:

$$\mathop{\mathbb{E}}_{\substack{v_{-i}\sim F_{-i},\\ b\sim s(V)}}[u_i(b)] \geq \frac{1}{2}v_i(OPT_i^{v^*}) - \mathop{\mathbb{E}}_{v_{-i},b_{-i}\sim s(v_{-i})}\left[\sum_{j\in OPT_i^{v^*}} \max_{k\neq i} b_k(j)\right]$$

Sum over all i and expectation over all $v_i \sim F_i$ and $v^*_{-i} \sim F_{-i}$:

$$\sum_{i\in N} \mathop{\mathbb{E}}_{\substack{v_i,v^*_{-i},\\ b\sim s(V)}}[u_i(b)] \geq \frac{1}{2}\sum_{i\in N} \mathop{\mathbb{E}}_{v,v^*_{-i}}\left[v_i(OPT_i^{v^*})\right] - \sum_{i\in N} \mathop{\mathbb{E}}_{v,v^*_{-i},b_{-i}\sim s(v_{-i})}\left[\sum_{j\in OPT_i^{v^*}} \max_{k\neq i} b_k(j)\right]$$

Proof of theorem 2

The LHS is the expected SW of s since v^*_{-i} does not appear in the expectation:

$$\sum_{i \in N} \mathbb{E}_{v, v^*_{-i} b \sim s(V)} [u_i(b)] = \mathbb{E}_{v, b \sim s(V)} \left[\sum_{i \in N} u_i(b) \right]$$

The first term in the RHS is half of the the expected optimum (relabeling):

$$\frac{1}{2} \sum_{i \in N} \mathbb{E}_{v_i, v^*_{-i}} \left[v_i(OPT_i^{v^*}) \right] = \frac{1}{2} \mathbb{E}_{v} \left[\sum_{i \in N} v_i(OPT_i^{v}) \right]$$

Proof of theorem 2

- Take a maximum over a larger set in the second term in the LHS:

$$\sum_{i \in N} \mathop{\mathbb{E}}_{\substack{v, v^*_{-i}, \\ b_{-i} \sim s(v_{-i})}} \left[\sum_{j \in OPT_i^{v^*}} \max_{k \neq i} b_k(j) \right] \leq \sum_{i \in N} \mathop{\mathbb{E}}_{\substack{v, v^*_{-i}, \widehat{v_i}, \\ b \sim s(\widehat{v_i}, v_{-i})}} \left[\sum_{j \in OPT_i^{v^*}} \max_{k} b_k(j) \right]$$

- $\widehat{v_i} \sim F_i$ is used to keep b independent of OPT^{v^*}.
- Since OPT^{v^*} imposes a partition on $[m]$ we get:

$$= \mathop{\mathbb{E}}_{v, b \sim s(v)} \left[\sum_{j \in [m]} \max_{k} b_k(j) \right]$$

Proof of theorem 2

In total:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} u_i(b)\right] \geq \frac{1}{2}\mathop{\mathbb{E}}_{v}\left[\sum_{i\in N} v_i(OPT_i^v)\right] - \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(v)}}\left[\sum_{j\in[m]} \max_k b_k(j)\right]$$

In first-price auction setting:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} u_i(b)\right] = \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} v_i(W_i(b))\right] - \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} \max_k b_k(j)\right]$$

Which concludes:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} v_i(W_i(b))\right] \geq \frac{1}{2}\mathop{\mathbb{E}}_{v}\left[\sum_{i\in N} v_i(OPT_i^v)\right]$$

This yields the desired result.

Proof of theorem 3

Let us now bound the BPoA for second-price item auctions:

- To reach the inequality:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} u_i(b)\right] \geq \frac{1}{2}\mathop{\mathbb{E}}_{v}\left[\sum_{i\in N} v_i(OPT_i^v)\right] - \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(v)}}\left[\sum_{j\in[m]} \max_k b_k(j)\right]$$

  Follow the same steps as before, using Lemma 3 instead of Lemma 2 to get a no-overbidding strategy.

- By dropping payments:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} v_i(W_i(b))\right] \geq \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} u_i(b)\right]$$

Proof of theorem 3

- Each item is assigned to the highest bidder:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(v)}}\left[\sum_{j\in[m]} \max_k b_k(j)\right] = \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(v)}}\left[\sum_{i\in N}\sum_{j\in W_i(b)} b_i(j)\right]$$

- Each player i is not-overbidding:

$$\leq \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(v)}}\left[\sum_{i\in N} v_i(W_i(b))\right]$$

- Plugging this to first inequality in the proof:

$$\mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(V)}}\left[\sum_{i\in N} v_i(W_i(b))\right] \geq \frac{1}{2}\mathop{\mathbb{E}}_{v}\left[\sum_{i\in N} v_i(OPT_i^v)\right] - \mathop{\mathbb{E}}_{\substack{v,\\ b\sim s(v)}}\left[\sum_{i\in N} v_i(W_i(b))\right]$$

Which results in BPoA ≤ 4

Lower bounds

If we assume strong no-overbidding:

- The BPoA of the simultaneous second price auction is strictly greater than 2
- If players' valuations may be correlated, the PoA is $\Omega(n^{1/4})$

If we assume strong no-overbidding:

- The BPoA of the simultaneous second price auction is strictly greater than 2
- If players' valuations may be correlated, the PoA is $\Omega(n^{1/6})$

Questions?

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