Strong Price of Anarchy

Price of anarchy seminar, with Prof. Michal Feldman

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Based on papers:

- *Strong price of anarchy. By N. Andelman, M. Feldman, and Y. Mansour.*
- *Strong equilibrium in cost sharing connection games. By Amir Epstein, Michal Feldman, Yishay Mansour.*

Motivation

- Selfish agent – optimizing its own utility rather than reaching the social optimum
- When considering the agent's incentives, Nash equilibrium is the most popular solution concept
- Is it a sustainable solution?
  - Nash equilibrium is resilient to unilateral deviations
  - Does not necessary imply that the solution is sustainable since other types of deviations might be possible

Strong Equilibrium (SE)

- Definition: **SE** is a state from which no coalition (of any size) can deviate and improve the utility of **every** member of the coalition (while possibly lowering the utility of players outside the coalition).
- Definition: **k-SE** is a state from which no coalition of at most k players can deviate and improve the utility of **every** member of the coalition.
- Note: Considering only pure equilibriums
- From the definitions:

$$SE = n - SE \subseteq (n-1) - SE \subseteq \cdots \subseteq 2 - SE \subseteq 1 - SE = NE$$

Strong Equilibrium (SE)

- Formal Definition:
- A **pure deviation** of a set of players $\Gamma \subseteq N$ (also called coalition) specifies an action for each player in the coalition, i.e., $\gamma \in \times_{i \in \Gamma} S_i$. A profile $s \in S$ is **not resilient to a pure deviation** of a coalition Γ if there is a pure joint action γ of Γ such that $c_i(s_{-\Gamma}, \gamma) < c_i(s)$ for every $i \in \Gamma$. A pure NE $s \in S$ is a **k-strong equilibrium**, if there is no coalition Γ of size at most k, such that s is not resilient to a pure deviation by Γ.

Strong Price of Anarchy (SPoA)

- Definitions: Similarly to PoA and PoS we define:
  - **SPoA** is the ratio of the worst SE and the optimum
  - **k-SPoA** is the ratio of the worst k-SE and the optimum
  - **SPoS** is the ratio of the best SE and the optimum
  - **k-SPoS** is the ratio of the best k-SE and the optimum

SPoA/SPoS – Formal Definitions

- Consider game G, and an objective function f_G
- The objective function is:
  - the social cost of $s \in S$ is $f_G(s)$
- The optimal social cost: $OPT(G) = min_{s \in S} f_G(s)$
- Let $\Phi(G, k)$ be the set of k-SE of the game G
- Definition: If $\Phi(G, k) \neq \emptyset$ then

$$k - SPoA = \frac{max_{s \in \Phi(G,k)} f_G(s)}{OPT(G)}$$

$$k - SPoS = \frac{min_{s \in \Phi(G,k)} f_G(s)}{OPT(G)}$$

SE and SPoA

- By definition: $SPoA \leq PoA$
- If SE exists, it may be very robust
- Major downside: Most games do not admit any SE
  - Even a simple classical game like Prisoner's dilemma
- The approach: to identify under what conditions (games and settings) the existence of a SE is guaranteed and measure its quality.
- We will discuss 3 models:
  - Job scheduling
  - Network creation
  - Cost sharing connection games

Job Scheduling - The Model

- $M = \{M_1, \dots, M_m\}$ the set of m machines
- $N = \{N_1, \dots, N_n\}$ the set of n players (jobs)
- $w_i(J)$ is the weight of player $J \in N$ on machine $M_i \in M$
- $S_J = M$ is the action space of player $J \in N$
- $S = S_1 \times \cdots \times S_n$ is the joint action space
- s_J - the machine player J selects in a joint action $s \in S$
- $B_{i,s} = \{J \mid s_J = M_i\}$ is the set of players on machine M_i
- $L_i(s) = \sum_{J \in B_{i,s}} w_i(J)$ is the load on machine M_i
- $c_J(s) = L_i(s)$ is the load that player J observes on M_i

Job Scheduling - The Model

- The social cost is the makespan
  - $makespan(s) = max_J c_J(s)$
- $OPT = min_s makespan(s)$
- We define:
  - $w_{min}(J) = min_i w_i(J)$
  - $\min(J) = argmin_i w_i(J)$
  - $OPT(J)$ is the action of job J under OPT

Job Scheduling – Equilibrium Existence

- <u>Definitions:</u>
  - A vector $(l_1, l_2, \ldots, l_m)$ is **smaller** than $(\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_m)$ **lexicographically** if for some $i, l_i < \hat{l}_i$ and $l_k = \hat{l}_k$ for all $k < i$.
  - A joint action s is smaller than s' lexicographically if the vector of machine loads $L(s)$, sorted in non increasing order, is smaller lexicographically than $L(s')$, sorted in non increasing order. We denote this by $\boldsymbol{s} \prec \boldsymbol{s}'$.

Job Scheduling – Equilibrium Existence

- Lemma 1: Consider two joint actions s and s' such that the load vectors $L(s)$ and $L(s')$ differ only in the loads of machines in a set $M' \subseteq M$. If for each $M_i \in M'$, $L_i(s) < max_k\{L_k(s')|M_k \in M'\}$ then $s \prec s'$.
- Lemma 2: The lexicographically minimal joint action s is NE
  - Assume J can benefit from deviating from $s_J = M_i$ to $s'_J = M_k$, then:
    - $L_k(s) < L_i(s)$
    - $L_i(s') < L_i(s)$
    - $L_k(s') < L_i(s)$
  - According to lemma 1, $s' \prec s$, and we get a contradiction.

Job Scheduling – Equilibrium Existence

- <u>Theorem:</u> In any job scheduling game, the lexicographically minimal joint action s is a k-SE, for any k.
- *Proof:*
  - Assume by contradiction that there is a coalition Γ of size $k \leq n$ that can deviate and lower the observed load of each player in it. Let Γ be minimal coalition.
  - Every job in the coalition must migrate (otherwise Γ is not minimal).
  - Let s' be the profile after the deviation
  - $M(\Gamma, \mathrm{s}) = \cup_{J \in \Gamma}\{s_J\}$
  - $M(\Gamma, \mathrm{s}) = M(\Gamma, \mathrm{s}')$

Job Scheduling – Equilibrium Existence

- Theorem: In any job scheduling game, the lexicographically minimal joint action s is a k-SE, for any k.
- *Proof cont.:*
  - $M' = M(\Gamma, s) = M(\Gamma, s')$
    - If there is a machine that a job migrates to but no job migrates from, it contradicts that s is a NE.
    - If there is a machine that a job migrates from but no job migrates to, it contradicts the minimality of Γ.
  - For each machine $M_i \in M'$ there is at least one job $J \in N$ that wishes to migrate to it and will benefit from it
    - ${s'}_J = M_i$ and $L_{{s'}_J}(s') < L_{s_J}(s)$

Job Scheduling – Equilibrium Existence

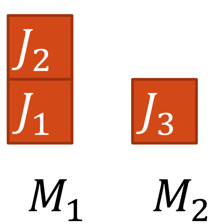
- Theorem: In any job scheduling game, the lexicographically minimal joint action s is a k-SE, for any k.
- *Proof cont.:*
  - We get that for each $M_i \in M'$:

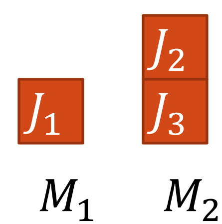
    $$L_i(s') < max_k\{L_k(s) | M_k \in M'\}$$

  - According to lemma 1, $s' \prec s$, and we get a contradiction to the minimality of s.
- Note: The theorem implies that k-SPoS is 1.

Job Scheduling – Equilibrium Existence

- The requirement that every member in a coalition strictly benefits from the deviation is a crucial assumption for the correctness of the Theorem.
  - If we relax the condition and require only that some member improves its cost and no other member of the coalition would lose from the deviation, there are job scheduling games that do not have any SE.
- Example: in 2 identical machines and 3 unit jobs no equilibrium is a 2-SE
- NE: Not 2-SE: $\Gamma = \{1,2\}$





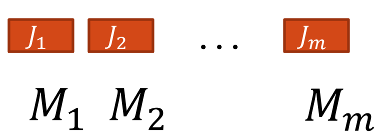
Job Scheduling – SPoA

- Identical machines:
  - $PoA \le 2$ and $SPoA \le PoA$.
  - The SPoA doesn't improve on the PoA, since $SPoA \ge \frac{2}{(1+\frac{1}{m})}$.
- Unrelated machines: The PoA is unbounded, does the SPoA improve on the PoA?
  - Example: consider $m \ge 2$ machines and $n = m$ jobs where

$$w_i(J_i) = \varepsilon \text{ for all } 1 \le i \le m \text{ and } w_i(J_j) = 1 \text{ for all } i \ne j$$

OPT: (also the only SE)

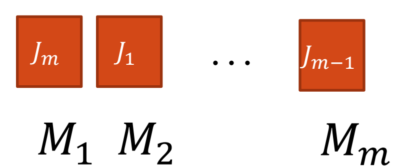
$makespan = \varepsilon$



$SPoA = 1$

NE:

$makespan = 1$



$PoA \ge \frac{1}{\varepsilon}$

Job Scheduling – SPoA

- For any job scheduling game with m unrelated machines and n jobs, the following bounds on the SPoA are proven:
  - $SPoA \leq 2m - 1$ and for $m = 2, SPoA \leq 2$
  - $SPoA \geq m$
  - The previous example also shows that $(m-1) - SPoA$ is unbounded.
  - For $k \geq m, SPoA \leq \frac{2nm}{z} + 4m$ where $z = \left\lfloor \frac{k}{m} \right\rfloor$
  - The worst-case SPoA is at most $O(nm^2/k)$ and at least $\Omega(n/k)$

Job Scheduling – SPoA

- Claim: For any job scheduling game with unrelated machines, the following inequalities hold:
  1) $OPT \geq max_J w_{min}(J)$
  2) $OPT \geq \frac{1}{m} \sum_J w_{min}(J)$
- Theorem: For any job scheduling game with 2 unrelated machines and n jobs, $SPoA \leq 2$.
  - *Proof:*
    - Let s be a SE and WLOG $L_2(s) \geq L_1(s)$
    - If for every $J \in B_{2,s}$, $w_2(J) \leq w_1(J)$ then by (2) we get $L_2(s) \leq 2OPT$.
    - Otherwise there exists $J \in B_{2,s}$ such that $w_2(J) > w_1(J)$

Job Scheduling – SPoA

- Theorem: For any job scheduling game with 2 unrelated machines and n jobs, $SPoA \leq 2$
  - *Proof cont.:*
    - There exists $J \in B_{2,s}$ such that $w_2(J) > w_1(J)$
    - s is a NE, hence $L_2(s) \leq L_1(s) + w_1(J)$
    - By (1) we get $L_2(s) \leq L_1(s) + OPT$
    - There are different cases regarding OPT:
      - $L_1(s) \leq L_2(s) < OPT$: contradicts the minimality of OPT
      - $OPT < L_1\ (s) \leq L_2(s)$: contradicts that s is SE (a coalition can deviate to OPT)
      - $L_1(s) \leq OPT \leq L_2(s)$: $L_2(s) \leq L_1(s) + OPT \leq 2OPT$
    - Taking the maximum over all cases, we get $SPoA \leq 2$

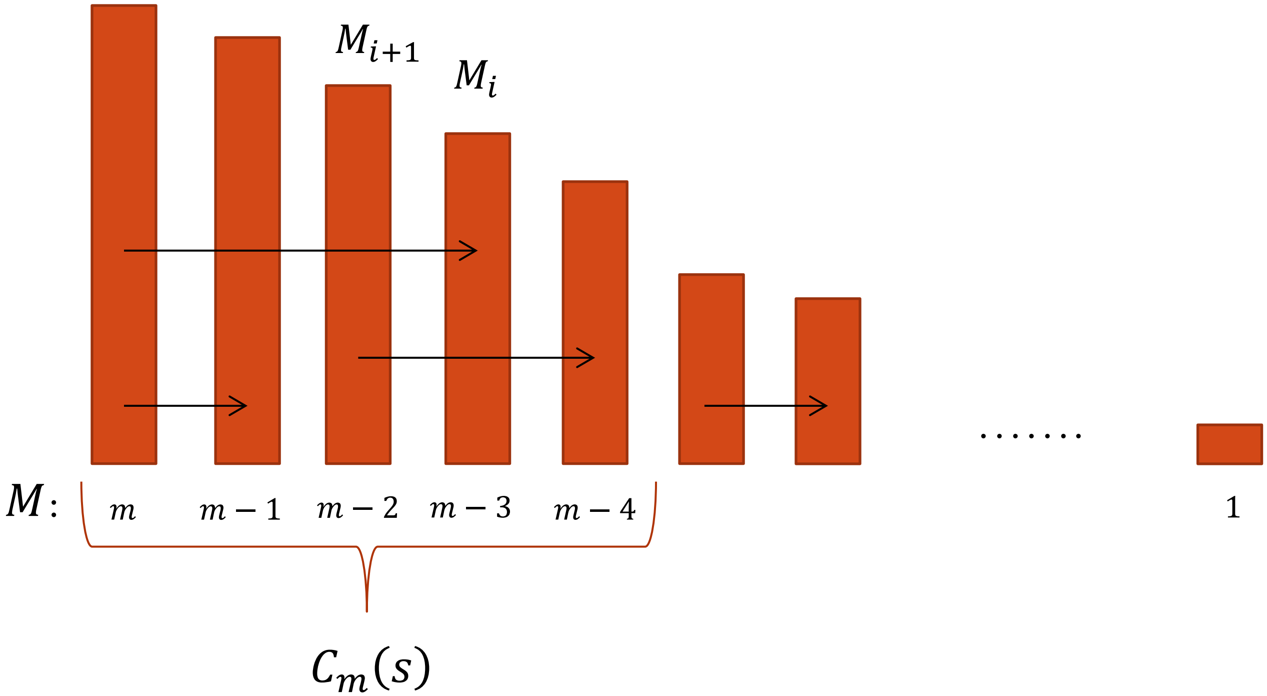
Job Scheduling – SPoA

- For the rest of this section we will assume WLOG that given a profile s, $L_1(s) \leq \cdots \leq L_m(s)$.
  - The machine indices are sorted in a non-decreasing order of the loads under s.
- Definition:
  - We denote by $M_i \underset{s}{\rightarrow} M_j$, if there is a job J such that $M_i = s_J$, $M_j = \min(J)$ and $i \geq j$.
  - M_i, M_j for $i \geq j$ are **connected** under the profile s if $\exists i', j'$ such that $i' \geq i \geq j \geq j'$, and $M_i \underset{s}{\rightarrow} M_j$.
  - $C_m(s) = \{M_m, \ldots, M_l\}$: the **maximal suffix** of machines, such that M_{i+1} is connected to M_i under the profile s.
- Claim: For every job J such that $s_J \in C_m(s)$ we have

$$\min(J) \in C_m(s).$$

Job Scheduling – SPoA

- $C_m(s) = \{M_m, \ldots, M_l\}$: the **maximal suffix** of machines, such that M_{i+1} is connected to M_i under the profile s.



Job Scheduling – SPoA

- Lemma: Let s be a NE. If $M_i \underset{s}{\rightarrow} M_j$ then:
  - $L_i(s) \leq L_j(s) + OPT$
  - $\forall i, j \in C_m(s)$: $L_i(s) \leq L_j(s) + (m-1)OPT$
- Proof:
  - $L_i(s) \leq L_j(s) + w_j(J)$ for each $J \in B_{i,s}$ (since s is NE)
  - $\exists J \in B_{i,s}$ $s.t. M_j = \min(J)$ (by definition of connectivity)
  - $w_j(J) \leq OPT$ (by inequality (1))
  - Overall $L_i(s) \leq L_j(s) + OPT$
  - By consecutive application of this argument the load of M_m and M_l differ at most by $(m-1)OPT$.

Job Scheduling – SPoA

- <u>Theorem</u>: For any job scheduling game with m unrelated machines and n jobs, $SPoA \le 2m - 1$.
- *Proof*:
  - Let s be a SE
  - If $\exists M_i \in C_m(s)\ s.t.\ L_i(s) \le m \cdot OPT$ then by the lemma we get $L_m(s) \le (2m-1)OPT$. Done.
  - Otherwise, $\forall M_i \in C_m(s), L_i(s) > m \cdot OPT$
    - Such profile s is not a SE: consider coalition $\Gamma = \bigcup_{M_i \in C_m(s)} B_{i,s}$
      - Profile s': $s'_J = \min(J)$ for $s_J \in C_m(s)$ (remember $\min(J) \in C_m(s)$)

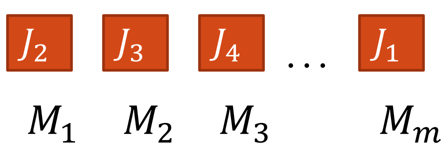
        $s'_J = s_J$ for $s_J \notin C_m(s)$
      - $\forall M_i \in C_m(s)$: $L_i(s') \le m \cdot OPT < L_i(s)$ (by inequality (1))
      - Overall we get that every job in the coalition is strictly better off under s'.

Job Scheduling – SPoA

- There exists a job scheduling game with m unrelated machines for which $SPoA \geq m$.
- *Proof:*
  - A game with m jobs and m unrelated machines
  - For $l = 2, \ldots, m$: $w_l(J_l) = l, w_{l-1}(J_l) = 1$ and $w_i(J_l) = \infty$ for $i \neq l, l-1$
  - For J_1: $w_1(J_1) = w_m(J_1) = 1$ and $w_i(J_1) = \infty$ for $i \neq 1, m$

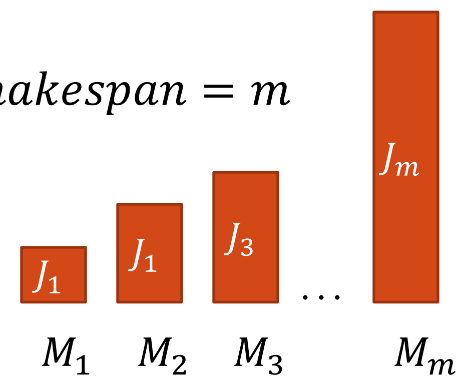
OPT:

$makespan = 1$



SE:

$makespan = m$



Network Creation – The Model

- $V = \{1, \ldots, n\}$ the set of players (vertices)
- $S_v = \{(v, u) | u \in V\{v\}\}$ is the action set
- Players buy edges at cost $\alpha > 0$ for an edge
- Given a profile $s = (s_1, \ldots, s_n), s_i \in S_i$, the resulting graph is $G = (V, E)$ when $E = \bigcup_{v \in V} s_v$
- $\delta_s(v, w)$ - the shortest path between v and w in $G(s)$
- $B_s(v) = \alpha|s_v|$ - the cost for the bought edges
- $Dist_s(v) = \sum_{w \in V} \delta_s(v, w)$ is the distance cost
- $c_v(s) = B_s(v) + Dist_s(v)$ - cost of a player
- The objective function is: $cost(s) = \sum_{v \in V} c_v(s)$
- The optimal cost is: $OPT = min_{s \in S} cost(s)$

Network Creation – Equilibrium Existence

- Theorem: Let s^* be a profile where $G(s^*)$ is a star in which all the nodes by edges to the root r. For $\alpha \geq 2$, the profile s^* is a SE.
- *Proof:*
  - Assume there exists a coalition Γ and a deviation s' such that each node in Γ strictly gains from deviating to s'.
  - $r \notin \Gamma$ since in s^* the root has the lowest possible cost
  - For any node $v \epsilon \Gamma$ denote:
    - x_v - the number of the new outgoing edges
    - y_v - the number of the new incoming edges
  - $\sum_{v \in \Gamma} x_v \geq \sum_{v \in \Gamma} y_v$ - since all new edges originated from Γ

Network Creation – Equilibrium Existence

- Theorem: Let s^* be a profile where $G(s^*)$ is a star in which all the nodes by edges to the root r. For $\alpha \geq 2$, the profile s^* is a SE.
- *Proof cont.:*
  - $\sum_{v \in \Gamma} x_v \geq \sum_{v \in \Gamma} y_v$:
  - Case 1: $\exists v \in \Gamma \text{ s.t. } x_v > y_v$
    - If v doesn't remove it's original edge to r, the changes in v's cost: $\alpha x_v - (x_v + y_v) \geq \alpha x_v - 2x_v + 1 \geq 0$
    - v removes it's original edge to r: the changes in v's cost: $\alpha x_v - (x_v + y_v) - \alpha + 1 \geq \alpha x_v - 2x_v + 2 - \alpha \geq (x_v - 1)(\alpha - 2) \geq 0$
    - v doesn't improve its cost, and we get a contradiction.
- Note: For $\alpha \geq 2$ a star is also the optimum, hence $SPoS = 1$

Network Creation – Equilibrium Existence

- Theorem: Let s^* be a profile where $G(s^*)$ is a star in which all the nodes by edges to the root r. For $\alpha \geq 2$, the profile s^* is a SE.
- *Proof cont.:*
  - $\sum_{v \in \Gamma} x_v \geq \sum_{v \in \Gamma} y_v$:
  - Case 2: $\forall v \in \Gamma: \quad x_v = y_v$
    - If v doesn't remove it's original edge to r, the changes in v's cost: $\alpha x_v - (x_v + y_v) = (\alpha - 2) x_v \geq 0$
    - The only way to improve the cost is to remove the edge to r, but then Γ won't be connected to the rest of the graph and the distance cost will be ∞.
    - v doesn't improve its cost, and we get a contradiction.

Network Creation – Equilibrium Existence

- Theorem: For $\alpha < 1, s$ is a SE iff $G(s)$ is a clique. For $\alpha = 1$, if $G(s)$ is a clique, then s is a SE.
- *Proof:*
  - For $\alpha < 1$ every NE is a clique, so if s is a SE then $G(s)$ is a clique.
  - The other direction holds for $\alpha \leq 1$:
    - s is a profile s.t. $G(s)$ is a clique
    - Γ is a coalition that deviates to s' and improves the cost of each node in it
    - $x = |E_{G(s)}| - |E_{G(s')}|$ - number of edges that are missing from the clique

Network Creation – Equilibrium Existence

- <u>Theorem</u>: For $\alpha < 1, s$ is a SE iff $G(s)$ is a clique. For $\alpha = 1$, if $G(s)$ is a clique, then s is a SE.
- *Proof:*
  - The other direction holds for $\alpha \leq 1$:
    - For each missing edge:
      - Exactly one $v \in \Gamma$ decreased $B(v)$ by α, thus $\sum_{v \in \Gamma} B(v)$ decreased by exactly $\alpha x \leq x$
      - At least one $v \in \Gamma$ increased his $Dist(v)$ by 1, thus $\sum_{v \in \Gamma} Dist(v)$ increased by at least x
    - The sum of the costs of the nodes in the coalition has not decreased
- Note: Since in this case a clique is the social optimum, the SPoS is 1

Network Creation – Equilibrium Existence

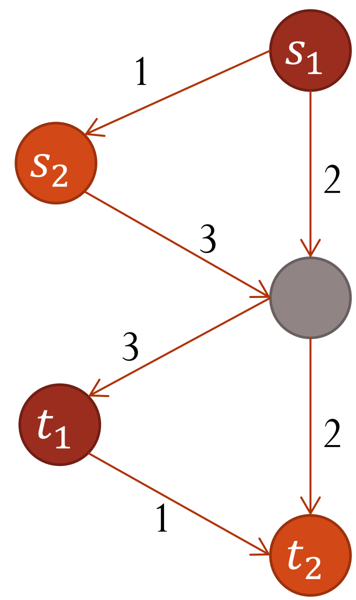
- More results:
- For $\alpha \in (1,2)$ there is no SE in general, even if we limit the coalition size to 3.
  - More specifically: for $n \geq 7$ there does not exist any 3-SE
  - The theorem holds also for $n = 6$
  - For $n = 5$ there does not exist any SE
  - For $n \leq 4$ there always exists a SE
  - The star graph is a 2-SE for any n

Network Creation – SPoA

- Theorem: For any $\alpha \geq 2$ and any n, we have $SPoA \leq 2$.
- Lemma 1: Let s be a NE. For every node v we have $cost(s) \leq (n-1)(2\alpha + n - 1 + Dist(v))$
- Lemma 2: Let s be a NE. Assume that for every node v, such that $s_v \neq \emptyset$, we have $Dist(v) > 3n - 5$. Then s is not a SE.
- Lemma 3: Let s be a NE. Assume that for some node v, such that $s_v \neq \emptyset$, we have $Dist(v) \leq 3n - 5$. Then $\frac{cost(s)}{cost(OPT)} \leq 2$
  - $cost(s) \leq (n-1)(2\alpha + n - 1 + 3n - 5)$
    $= 2(n-1)(\alpha + 2n - 3)$
  - $cost(OPT) = \alpha(n-1) + (n-1)(2(n-2)+1) + (n-1)$
    $= (n-1)(\alpha + 2n - 2)$

Cost Sharing Connection Games

- A network design game in which:
  - there is an underlying directed graph with edge costs
  - each of n users has a specified source node and sink node he wishes to connect, while paying as little as possible.
  - The cost for each user is determined by some predefined sharing rule
    - fair connection game - cost of an edge is divided equally among all the users that have it in their chosen path
    - general connection game - each user offers prices for the edges he wants to use according to his chosen path, and an edge is bought if the sum of all the offers for this edge covers its cost



Cost Sharing Connection Games – The Model

- $G = (V, E)$ – an underlying directed graph
- Each $e \in E$ has a finite cost $c_e \geq 0$
- $N = \{N_1, \dots, N_n\}$ the set of n players
- Each player $i \in N$ has source s_i and sink t_i
- Fair connection game:
  - The action S_i includes all the paths from s_i to t_i
  - $n_e(s)$ – number of players selected a path containing e in s
  - The cost function of each player on edge e:

    $$f_e(n_e(s)) = \frac{c_e}{n_e(s)}$$

  - The cost of player i, when selecting path $Q_i \in S_i$:

    $$c_i(s) = \sum_{e \epsilon Q_i} f_e(n_e(s))$$

Cost Sharing Connection Games – The Model

- General connection game:
  - The action S_i is a payment vector p_i
  - $p_i(e)$ – how much player i is willing to contribute to buy e
  - For profile $p = (p_1, p_2, \dots, p_n)$, E_p is the set of bought edges
  - $e \in E_p$ if $\sum_i p_i(e) \geq c_e$
  - $G_p = (V, E_p)$ – the resulting graph
  - The total cost of a player is $c_i(p) = \sum_{e \in E_p} p_i(e)$ if s_i is connected to t_i in G_p, and ∞ otherwise.
  - The total cost for profile p: $c(p) = \sum_i c_i(p)$

Cost Sharing Connection Games

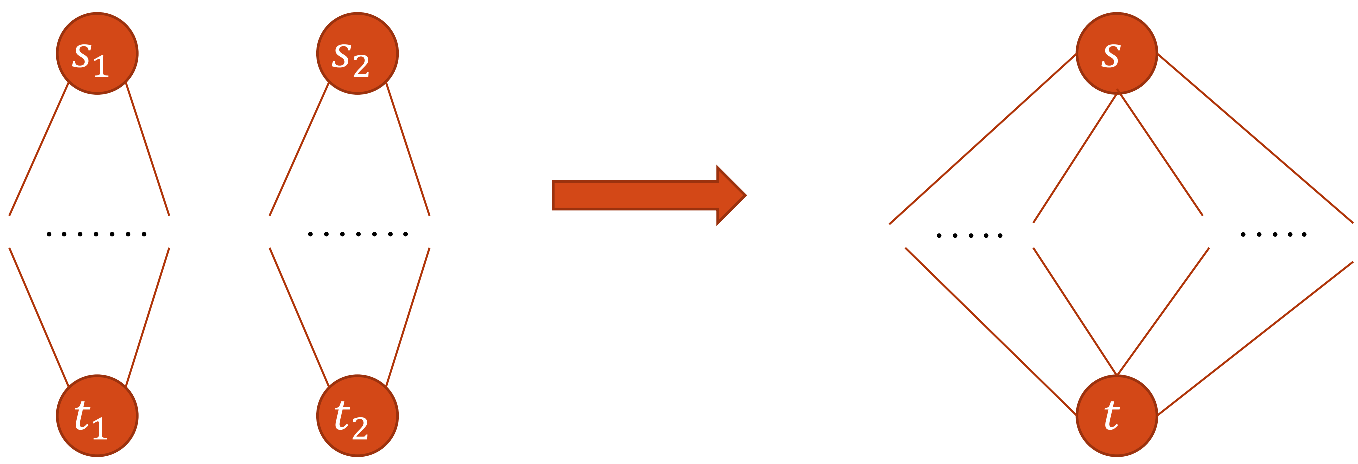
- We consider 3 types of games, regarding the source and sink:
  1) **Symmetric** connection game- all players share the same source and sink nodes
  2) **Single source** connection game – all players share the same source
  3) **Multi commodity** connection game – each player has its own source and sink

Cost Sharing Connection Games

- **Goal**: Identify under what conditions the existence of a SE is guaranteed.
  - Our underlying directed graph would be acyclic, with source and sink nodes
    - Source node – can reach every node
    - Sink node – can be reached by every node
  - We will discuss 2 families of acyclic graphs:
    - Extension parallel graph
    - Series parallel graph

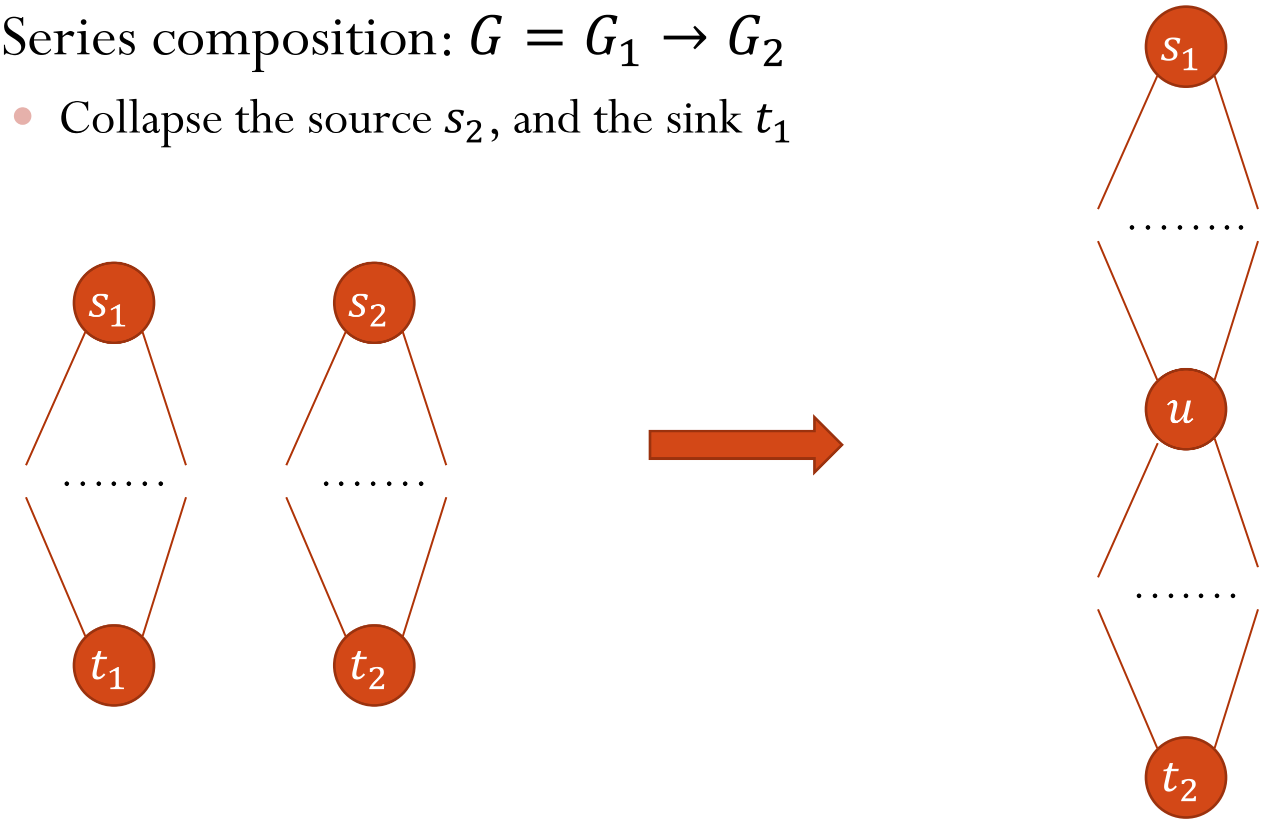
Cost Sharing Connection Games

- Given 2 directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1, s_2 \in V_2$ and sinks $t_1 \in V_1, t_2 \in V_2$ we define the following actions for composition:
  - Parallel composition: $G = G_1 || G_2$
    - Collapse the sources to one sources s, and the sinks to one sink t



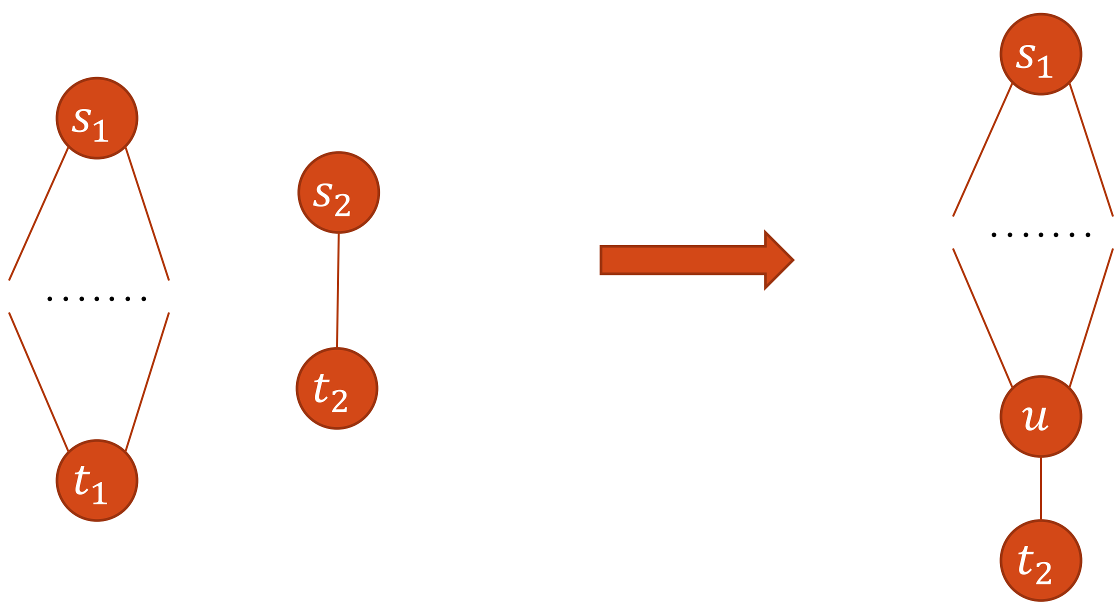
Cost Sharing Connection Games

- Given 2 directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1, s_2 \in V_2$ and sinks $t_1 \in V_1, t_2 \in V_2$ we define the following actions for composition:
  - Series composition: $G = G_1 \rightarrow G_2$
    - Collapse the source s_2, and the sink t_1



Cost Sharing Connection Games

- Given 2 directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1, s_2 \in V_2$ and sinks $t_1 \in V_1, t_2 \in V_2$ we define the following actions for composition:
  - Extension composition: $G = G_1 \rightarrow_e G_2$
    - A series composition when one of the graphs is a single directed edge



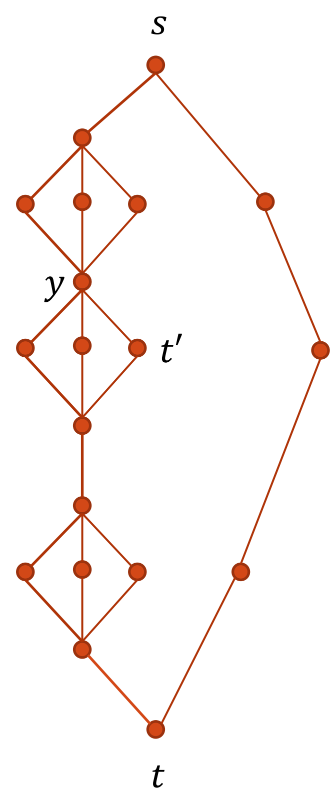
Cost Sharing Connection Games

- **Extension parallel graph (EPG)**
  1. A single directed edge
  2. A graph $G = G_1 || G_2$
  3. A graph $G = G_1 \rightarrow_e G_2$
     - Where G_1 and G_2 are extension parallel graphs.
- **Series parallel graph (SPG)**
  1. A single directed edge
  2. A graph $G = G_1 || G_2$
  3. A graph $G = G_1 \rightarrow G_2$
     - Where G_1 and G_2 are series parallel graphs.

Cost Sharing Connection Games

- Lemma:

  Let G be an SPG with source s and sink t.
  Given a path Q from s to t,
  and a vertex t',
  there exists a vertex $y \in Q$,
  such that for any path Q' from s to t',
  the path Q' contains y and
  the paths $Q'_{y,t'}$ and Q are edge disjoint.
  We call the vertex y
  the intersecting vertex of Q and t'.



Fair Connection Games Equilibrium Existence

- Theorem: Every symmetric fair cost sharing connection game admits a strong equilibrium.
- *Proof*:
  - s', t' - the source and sink nodes of all the players, respectively
  - Profile s – all the players choose the shortest path Q
  - Suppose s is not a SE:
    - There is a coalition Γ that can deviate to a new profile s' and improve the cost of every player
    - Let ${Q'}_j$ be a new path used by $j \in \Gamma$
    - $c({Q'}_j \backslash (Q \cap {Q'}_j)) \geq c(Q_j \backslash (Q \cap {Q'}_j))$ – since Q is the shortest path
    - Therefore, for every $j \in \Gamma:\ c_j(s') \geq c_j(s)$ - contradiction

Fair Connection Games Equilibrium Existence

- The minimality lemma:
- Let Λ be a fair connection game on a SPG G with source s and sink t. Assume that player i has $s_i = s, t_i = t$ and that Λ has some SE.
- Let s be a SE that minimizes the cost of player i

$$c_i(s) = min_{T \in SE} c_i(T)$$

- Let s^* be the profile that minimizes the cost of player i

$$c_i(s^*) = min_{T \in S} c_i(T)$$

- Then, $c_i(s) = c_i(s^*)$

Fair Connection Games Equilibrium Existence

- Lemma: Let Λ be a fair connection game on a graph $G = G_1||G_2$, where G_1 and G_2 are SPGs. If every fair connection game on the graphs G_1 and G_2 possesses a SE, then the game Λ possesses a SE.
- *Proof*:
  - Let graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources s_1,s_2 and sinks, t_1, t_2 respectively.
  - T_i - the set of players with an endpoint in $V_i \backslash \{s, t\}$ $(i = 1,2)$
  - T_3 - the set of players with source s and sink t (of G)
  - Note: $V_1 = T_1 \cup T_3$ and $V_2 = T_2 \cup T_3$
  - Λ_1, Λ_2 - the original games on G_1, G_2, respectively

Fair Connection Games Equilibrium Existence

- Lemma: Let Λ be a fair connection game on a graph $G = G_1||G_2$, where G_1 and G_2 are SPGs. If every fair connection game on the graphs G_1 and G_2 possesses a SE, then the game Λ possesses a SE.
- *Proof*:
  - s', s'' - SE in Λ_1, Λ_2 that minimizes the cost of players in T_3
  - WLOG: $c_i(s') \leq c_i(s'')$ where $i \in T_3$
    - Note: in SE, all players in T_3 follow the same path and have the same cost
  - Λ'_2 - the game on G_2 with players T_2, and $\bar{s}$ be a SE in Λ'_2
  - The profile $s = s' \cup \bar{s}$ is a SE
  - Otherwise there is a coalition Γ that can deviate and decrease the cost of each of its players

Fair Connection Games Equilibrium Existence

- Lemma: Let Λ be a fair connection game on a graph $G = G_1||G_2$, where G_1 and G_2 are SPGs. If every fair connection game on the graphs G_1 and G_2 possesses a SE, then the game Λ possesses a SE.
- *Proof*:
  - By the minimality lemma and the assumption $c_i(s') \leq c_i(s'')$ a player in T_3 cannot improve its cost.
  - Therefore: $\Gamma \subseteq T_1 \cup T_2$
  - Contradiction to s' being SE in Λ_1 or $\bar{s}$ being a SE in Λ'_2

Fair Connection Games Equilibrium Existence

- Theorem: Every single source fair connection game on a series parallel graph possesses a SE.
- Theorem: Every fair connection game on an extension parallel graph possesses a SE.

For more general topologies:

- Theorem:
  - There exists a multi commodity fair connection game on a SPG (that is not EPG) that does not possess a SE.
  - There exists a single source fair connection game that does not possesses a SE.

Fair Connection Games Equilibrium Existence

- <u>Theorem</u>: Every single source fair connection game on a SPG possesses a SE.
- *Proof*:
  - By induction on the network size $|V|$:
  - $|V| = 2$: obvious
  - Let graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources s_1,s_2 and sinks, t_1, t_2 respectively.
  - We show the claim for a series composition $G = G_1 \rightarrow G_2$ and for a parallel composition $G = G_1 || G_2$.
  - Parallel composition: follows directly from the previous lemma.

Fair Connection Games Equilibrium Existence

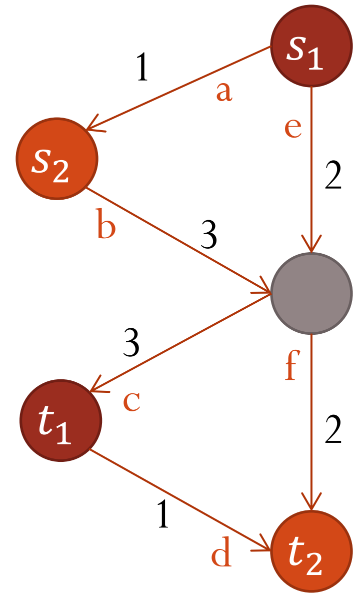
- <u>Theorem</u>: Every single source fair connection game on a SPG possesses a SE.
- *Proof*:
  - Series composition: $G = G_1 \to G_2$
  - T_1 - the set of players with sink in V_1
  - T_2 - the set of players with sink in $V_2 \setminus \{s_2\}$
  - Λ_1, Λ_2 - the original games on G_1, G_2, with players $T_1 \cup T_2, T_2$ respectively
  - For player $i \in T_2$ with action s_i in Λ :
    - $s_i \cap E_1$ is the action in Λ_1 (the sinks of all players in T_2 are set to t_1)
    - $s_i \cap E_2$ is the action in Λ_2 (the sources of all players in T_2 are set to s_2)

Fair Connection Games
Equilibrium Existence

- Theorem: Every single source fair connection game on a SPG possesses a SE.
- *Proof*:
  - Let s' be a SE in Λ_1 that minimizes the cost of players in T_2.
    - By the induction hypothesis and the minimality lemma
  - Let s'' be a SE in Λ_2
  - $s = s' \cup s''$ is a SE in the game Λ:
    - Assume a coalition Γ can deviate and improve the cost of every member:
      - $\Gamma \subseteq T_1$: contradiction to s' being a SE
      - $\exists j \in \Gamma \cap T_2$: by the minimality lemma player j cannot improve its cost in Λ_1, so the improvement must be in Λ_2. But then the coalition $\Gamma \cap T_2$ contradicts that s'' is a SE.

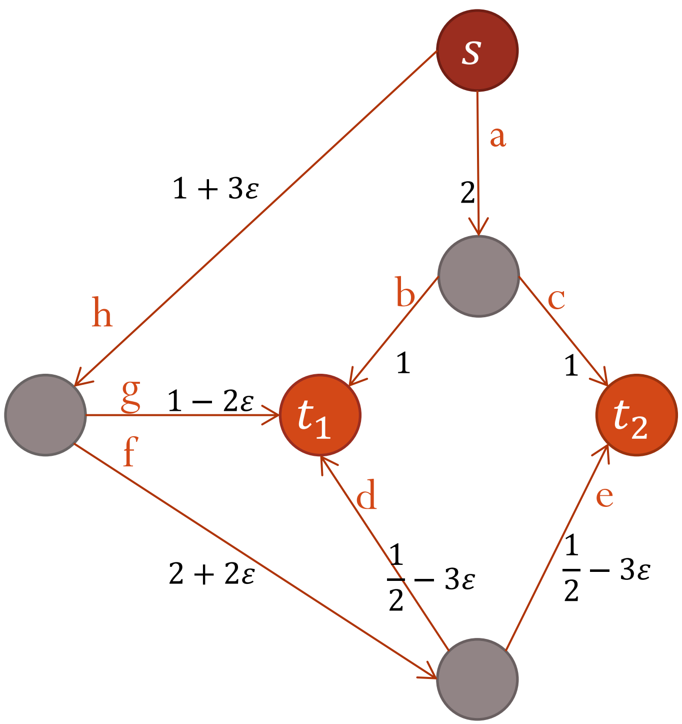
Fair Connection Games
Equilibrium Existence

- <u>Theorem</u>:
  - There exists a multi commodity fair connection game on a SPG (that is not EPG) that does not possess a SE.
  - *Proof*:
    - Unique NE s:
      - $s_1 = \{e, c\}$, $s_1 = \{\mathrm{b}, f\}$
        - Since the alternate paths for e and f will cost 2.5
      - Cost 5 for each player
    - Not SE:
      - Coalition Γ of both players, can deviate to s':
      - $s'_1 = \{a, b, c\}$, $s'_2 = \{b, c, d\}$
      - Cost 4 for each player



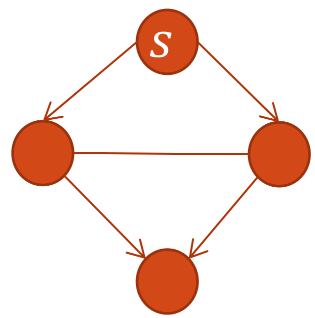
Fair Connection Games Equilibrium Existence

- <u>Theorem</u>:
  - There exists a single source fair connection game that does not possesses a SE.
  - *Proof*:
    - Unique NE s:
      - $s_1 = \{a, b\}, \; s_1 = \{a, c\}$
        - Always the best response
      - Cost 2 for each player
    - Not SE:
      - Coalition Γ of both players, can deviate to s':
      - $s'_1 = \{h, f, d\}, s'_2 = \{h, f, e\}$
      - Cost $2 - \frac{\varepsilon}{2}$ for each player

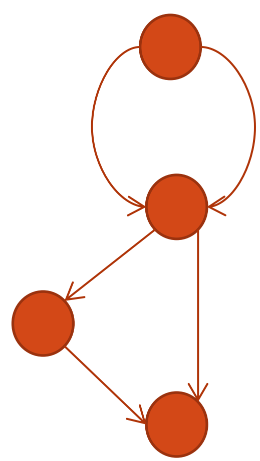


Fair Connection Games Equilibrium Existence

- The topological conditions are sufficient but not necessary:
  - Non-series parallel graph with a single source that always admits a SE:



  - Non-extension parallel graph that always admits a SE:



Fair Connection Games
SPoA – Upper Bound

- Theorem: The strong price of anarchy of a fair connection game with n players is at most $H(n)$
- *Proof*:
  - Let s be a SE and s^* be the OPT.
  - Let s_Γ be the induced profile of s of players in Γ.
  - Since s is a SE, there exists a player in $\Gamma_k = \{1, \ldots, k\}$, for $k = n, \ldots, 1$ (WLOG call him k), such that:
  $$c_k(s) \leq c_k(s_{-\Gamma_k}, {s^*}_{\Gamma_k}) \leq c_k({s^*}_{\Gamma_k})$$
  - The potential function:
  $$\Phi(s) = \sum_e c_e \sum_{i=1}^{n_e(s)} \frac{1}{i}$$

Fair Connection Games SPoA – Upper Bound

- Theorem: The strong price of anarchy of a fair connection game with n players is at most $H(n)$
- *Proof*:
  - Claim: $c_k\left(s^*{}_{\Gamma_k}\right) = \Phi\left(s^*{}_{\Gamma_k}\right) - \Phi\left(s^*{}_{\Gamma_{k-1}}\right)$
  - Summing over all player we get:

$$
\begin{aligned}
\sum_{i \in N} c_i(s) &\leq \sum_{i \in N} \Phi\left(s^*{}_{\Gamma_i}\right) - \Phi\left(s^*{}_{\Gamma_{i-1}}\right) \\
&= \Phi\left(s^*{}_{\Gamma_n}\right) - \Phi(\emptyset) \\
&= \Phi(s^*) \\
&= \sum_{e \in s^*} c_e H(n_e(s^*)) \\
&= H(n) OPT
\end{aligned}
$$

Fair Connection Games SPoA – Upper Bound

- Theorem: The SPoA of a fair connection game with non-decreasing concave edge cost functions and n players is at most $H(n)$.
- *Proof:*
  - Let be $c_e(n_e)$ be the cost function
    - Since it's concave, the cost per an edge might increase with the number of players, but the cost per player $\frac{c_e(n_e)}{n_e}$ decreases.
  - The key inequalities from the previous proof holds:

$$cost(s) \underbrace{\leq}_{1} \Phi(s^*) \underbrace{\leq}_{2} H(n)OPT$$

1) $c_k(s) \leq c_k(s_{-\Gamma_k}, {s^*}_{\Gamma_k}) \leq c_k({s^*}_{\Gamma_k})$
2) $c_e(x)$ non decreasing, hence: $\sum_{x=1}^{n_e} \frac{c_e(x)}{x} \leq H(n_e)c_e(n_e)$

Fair Connection Games SPoA – Upper Bound

- Theorem: The k-SPoA of a fair connection game with n players is at most $\frac{n}{k}H(k)$.
- *Proof*:
  - Let s be a SE and s^* be the OPT.
  - For simplicity assume n/k is an integer.
  - $T_1, \ldots, T_{n/k}$ - partition of the players to groups of size k.
  - Λ_j - the induced game by profile s, when T_j play s_{T_j} and $N \backslash T_j$ is fixed to s_{-T_j}.
  - The obtained edge cost function in Λ_j is non-decreasing and concave, hence the previous theorem holds:

Fair Connection Games SPoA – Upper Bound

- Theorem: The k-SPoA of a fair connection game with n players is at most $\frac{n}{k} H(k)$.
- *Proof*:
  - According to previous theorem:

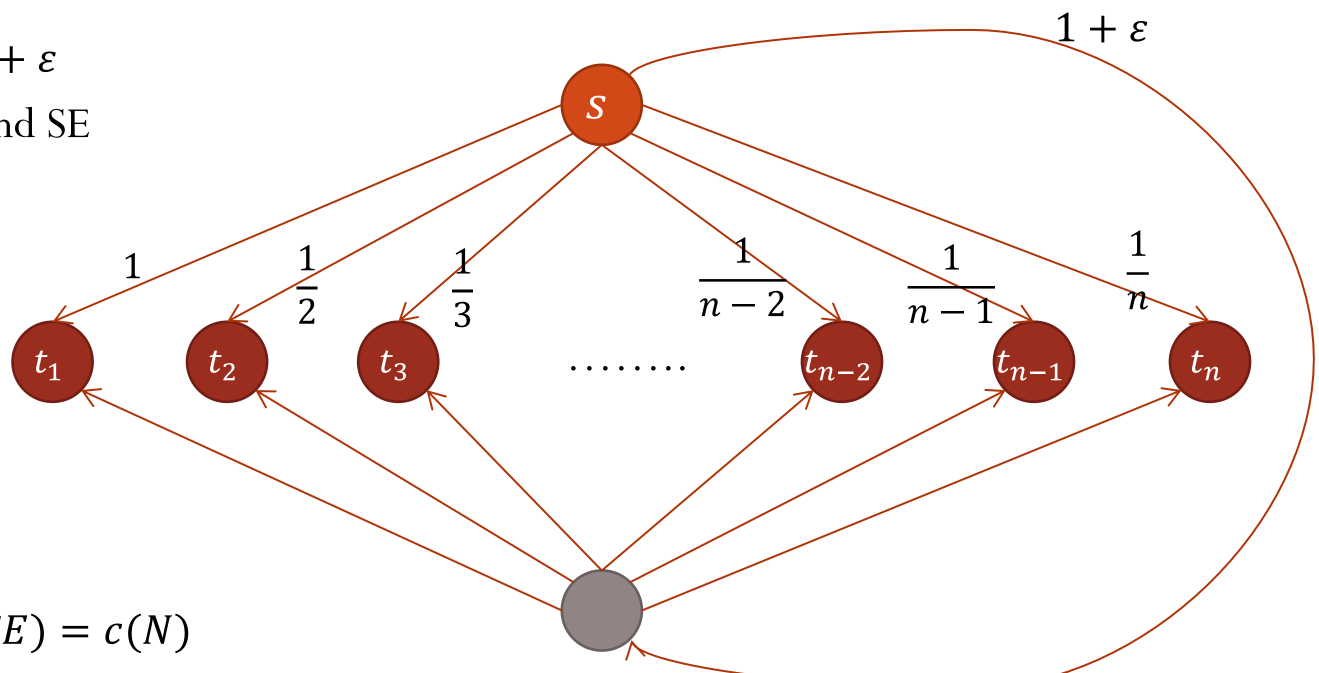
$$cost_{\Lambda_j}\left(s_{T_j}\right) = \sum_{i \in T_j} c_i(s) \leq H(k) OPT(\Lambda_j) \leq H(K) OPT(\Lambda)$$

  - Summing over all Λ_j:

$$cost_{\Lambda}(s) = \sum_{j=1}^{n/k} cost_{\Lambda_j}(s_{T_j}) \leq \frac{n}{k} H(k) OPT(\Lambda)$$

Fair Connection Games
SPoA – Lower Bound

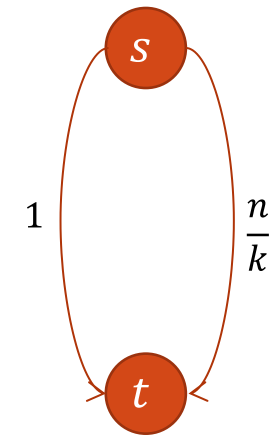
- Theorem: For fair connection game with n players, $k - SPoA \geq \max\{n/k, H(n)\}$
- *Proof*:
  - $c(OPT) = 1 + \varepsilon$
  - Unique NE and SE



  - $c(NE) = c(SE) = c(N)$
  - $PoS = SPoS = SPoA = PoA = H(n)$

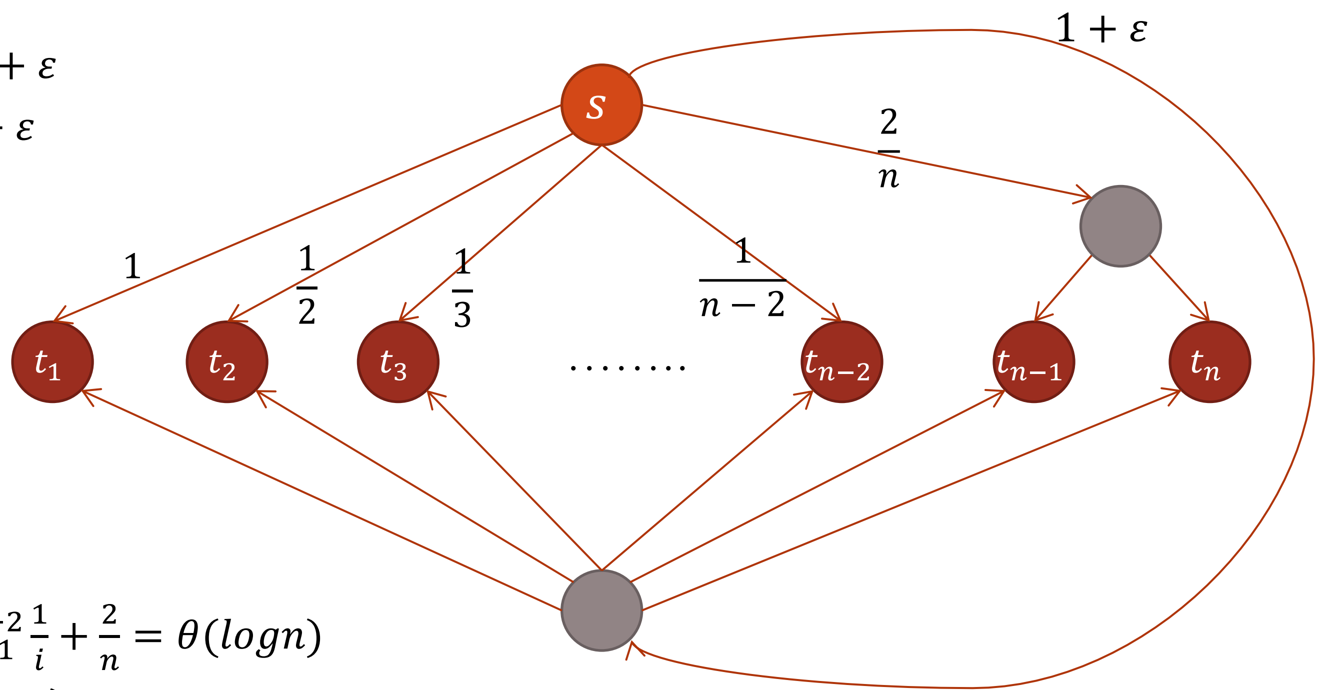
Fair Connection Games
SPoA – Lower Bound

- <u>Theorem</u>: For fair connection game with n players, $k - SPoA \geq \max\{n/k, H(n)\}$
- *Proof*:
  - $c(OPT) = 1$
  - $c(k - SE) = \frac{n}{k}$
    - All players choose the edge with cost $\frac{n}{k}$:
      - Cost of each player is $\frac{1}{k}$
      - If coalition up to size k will deviate to edge with cost 1, the cost of each player is at least $\frac{1}{k}$
  - $k - SpoA = \frac{n}{k}$



Fair Connection Games
SPoS

- <u>Theorem</u>: There exist a fair connection game in which $SPoS > PoS$.
- *Proof*:
  - $c(OPT) = 1 + \varepsilon$
  - $c(NE) = 1 + \varepsilon$
  - $PoS = 1$



  - $c(SE) = \sum_{i=1}^{n-2} \frac{1}{i} + \frac{2}{n} = \theta(logn)$
  - $SPoS = \theta(logn)$

Fair Connection Games
SPoS

- Note: The known upper bound for PoS still holds, since $SPoS \leq SPoA \leq H(n)$

- Overall we get:

$$PoS < SPoS \leq SPoA \leq H(n)$$

General Connection Games Equilibrium Existence

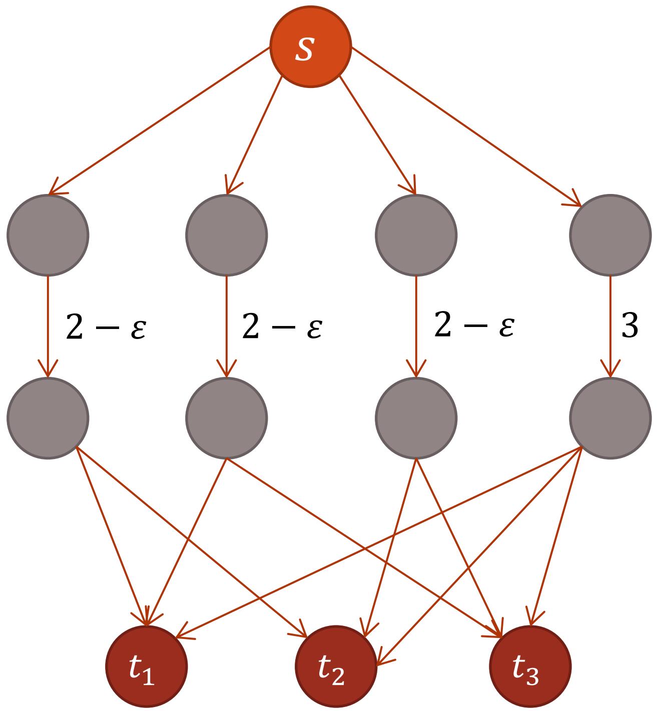
- The Model:
  - The action S_i is a payment vector p_i
  - $p_i(e)$ – how much player i is willing to contribute to buy e
  - For profile $p = (p_1, p_2, \dots, p_n)$, E_p is the set of bought edges
  - $e \in E_p$ if $\sum_i p_i(e) \geq c_e$
  - $G_p = (V, E_p)$ – the resulting graph
  - The total cost of a player is $c_i(p) = \sum_{e \in E_p} p_i(e)$ if s_i is connected to t_i in G_p, and ∞ otherwise.
  - The total cost for profile p: $c(p) = \sum_i c_i(p)$

General Connection Games Equilibrium Existence

- Theorem: In every symmetric general connection game there exists a SE.
- *Proof*: as in fair connection games
- Theorem: There exists a single source general connection game that does not admit any SE.
- Theorem: Every single source general connection game on a series parallel graph admits a SE.
  - Define an order on the sinks, such that a lower indexed sink has a path to the higher indexed sink.
  - Each player $j > i$ will buy the shortest path from i to j:
    - pay c_e for every edge.
    - The cost of all edges "before" i is set to zero, so the payment for those edges will be zero.
  - This algorithm results in a SE.

General Connection Games Equilibrium Existence

- <u>Theorem</u>: There exists a single source general connection game that does not admit any SE.
- *Proof*:
  - Check every NE:
    1. All players use 3:
       - There are 2 agents that pay together more than 2
       - They can move and pay $2 - \epsilon$
    2. 2 players use the same $2 - \epsilon$, and the third uses different $2 - \epsilon$ edge.
       - The players with $p \neq 0$ can move to edge 3.
       - Their overall payment will reduce to 3 from $4 - 2\varepsilon$



General Connection Games
SPoA

- Results for the SPoA (without proof):
  - Theorem: In single source general connection game, if there exists a SE then $SPoA = 1$ (it is the optimum).
  - Theorem: In any multi commodity general connection game, if there exists a SE then $SPoA = 1$ (it is the optimum).

Cost Sharing Connection Games

	Fair Connection	General Connection
single source single sink	always	always
single source multiple sinks	SPG not always	SPG not always
multiple sources multiple sinks	EPG not on SPG	