Algorithmic Game Theory - Fall 2013

Lecturer: *Michal* *Feldman*

Last Update: 29 Oct 2013 10:14 p.m. *Deadline: November 13, 2013*

Problem set 1

1. **[dominated strategies]** This question concerns a game between two players: Alice who has a finite strategy set S and Bob who has a finite strategy set T. We will denote Alice's utility when she plays s and Bob plays t by $a(s,t)$ and Bob's utility by $b(s,t)$. We start with the following definitions of strict dominance:

   - s is *purely-dominated* if for some $s' \in S$ we have that for all $t \in T$: $a(s,t) < a(s',t)$.
   - s is *mixed-dominated* if for some $x \in \Delta(S)$ we have that for all $t \in T$: $a(s,t) < a(x,t)$.
   - s is *never-best-reply* if for every $t \in T$ there exists $s' \in S$ so that: $a(s,t) < a(s',t)$.
   - Similar, dual, notions apply for the strategies of Bob.

   (a) Prove: s is purely-dominated $\Rightarrow$ s is mixed-dominated $\Rightarrow$ s is never-best-reply.

   (b) Give examples showing that the opposite implications are false.

   (c) Show that all three notions can be computed in polynomial time.

   (d) We could have also defined s to be a *never-best-reply-to-mixed* if for every $y \in \Delta(T)$ there exists $s' \in S$ so that: $a(s,y) < a(s',y)$. Prove that this is equivalent to s being mixed- dominated. (Hint: define a zero-sum game using Alice's utility.)

2. **[Mixed NE in two-player games]**

   (a) Show that the following problem can be solved in polynomial time: Given a two-player game, a subset of the rows S and a subset of the columns T, find a mixed-Nash equilibrium where the support of the row strategy is exactly S and the support of the column strategy exactly T , or state that such an equilibrium does not exists.

   (b) Show that there exists an exponential time algorithm for computing a mixed-Nash equilibrium in two-player games.

3. **[Potential games]** A *better reply sequence* in an n-player game G is a sequence (finite or infinite) of strategy profiles $s^0, s^1, \ldots, s^t, \ldots$ where s^0 is arbitrary, and for each $t > 0$, s^t is obtained from S^{t-1} by a single player i switching his strategy to a better one. I.e., for each $t > 0$ there exists i such that $s^t_{-i} = s^{t-1}_{-i}$ and $u_i(s^t) > u_i(s^{t-1})$. We say that all better reply dynamics converge on G if there are no infinite better reply sequences.

   (a) Prove that G is an ordinal potential game if and only if all better reply dynamics on G converge.

   (b) Show that there exists an ordinal potential game with n players, where each player has only two strategies, that has better-reply sequences of length 2^n.

4. **[Load balancing]** This question deals with the following bandwidth selection game: There are n radio users, where each user i needs to transmit q_i bits of information. There are two available radio bands, and each radio user can choose which band to use. The radios that use a certain band b share the bandwidth of the band, and all of them finish together at time. The two bands are equivalent for all purposes and the only thing that the users want is to finish as early as possible.

(a) Formalize this as a game.

(b) Show that the allocation of users to bands that minimizes the makespan (the load on the most loaded band) is a pure Nash equilibrium.

(c) Give an example where there is a pure equilibrium that does not minimize the makespan.

(d) Prove that if we start from an arbitrary partition of the users to the two bands and then repeat the following step sufficiently many times (but only finitely many times), we reach a pure Nash equilibrium. Step: take an arbitrary user that can improve his utility by moving to the other band and move him there.

(e) Give an example where this process takes an exponential (in n) number of steps until it reaches equilibrium.

(f) Show that finding the allocation that minimizes the makespan is NP-complete (hint: reduction from partition.)

(g) Give a polynomial time algorithm to find a pure Nash equilibrium.

(h) Assume now that, as opposed to all previous parts of the question, each user may partition his radio transmission between the two bands (e.g. sending $q_i/3$ in band 1 and $2q_i/3$ in band 2). Find strategies for the users that (1) are in equilibrium (2) together minimize the makespan (3) each user can decide what to do without looking at the others loads.

5. **[Selfish routing]** Prove that in an atomic selfish routing network of parallel links, every equilibrium flow minimizes the potential function.